译 者 介 绍

齐民友：安徽芜湖人．武汉大学数学系教授．1952 年毕业于武汉大学数学系．历任武汉大学研究生院院长、副校长、校长．曾为国务院学位委员会第二届数学学科评议组成员，中国数学学会第三届副理事长，湖北省数学学会理事长，全国人大文教委员会委员．是我国偏微分方程领域内的著名学者．

路见可：江苏宜兴人．武汉大学数学系教授．1943 年毕业于武汉大学数学系．历任武汉大学数学系主任、数学研究所所长等职，是我国第一批授予的博士生导师之一．曾为中国数学学会常务理事，现为湖北省数学学会和武汉数学学会理事长．主编过《数学杂志》，为《数学年刊》、《数学物理学报》等著名杂志编委．国际上受聘于美国、前西德《数学评论》杂志为特约评论员．精研函数论，是我国边值问题研究前沿中卓越的学术领导人．

中译本序

这个译本第一次问世是在 1981 年，在经历了 1/4 个世纪后再次与我国读者见面，是很有意义的．

第一次与读者见面时，正是改革开放初期，大家对获取新的数学知识有着极高的热情．而现在再次呈献给读者时，人们关注的仍然是如何使我国的数学教育与研究工作更好地跟上世界数学科学发展的步伐．从不少读过或教过此书的读者的反映看来，这本书仍是有益的．正如原书序言说的，本书内容是初等的，但是探讨的方法都是现代的．它与我们常见的经典的微积分教材比较，具有明显的特色“现代的和经典的处理方式按照完全不同的思路进行．其间有许多交汇点，最终汇合在最后一节．”读了这本书后，后续的读物是什么？也如原书序言所说：“至少有一半主要的数学分支都可以很有根据地推荐为本书内容的合理的继续”．由此可以看到这本书所介绍的内容，特别是处理方法对读者在数学上的发展有着非常重要的价值．

那么，读这本书是不是很难？这要看读者的要求．本书篇幅少，内容简洁，陈述也不艰深．如果只是粗略读一次，至少能学会现代数学的某些概念，用语和方法．但是真正的问题在于，现代的与经典的数学比较，在思路、风格上都大有不同．要想学到一些现代数学的思想与方法，进而能运用自如，当然不是易事．所以原书作者希望读者“鼓起勇气彻底学好第 4 章，确信花的工夫是值得的”．数学界有一句“格言”：数学不是看懂的，而是算懂的．意思是想要真正掌握数学，惟一的办法就是拿起笔来自己算上一算．所以原书序言说“习题是本书最重要的部分”．当本书责任编辑建议为新的本子作一个习题解答时，我们开始还有一些犹豫．因为不少同志都说，一本好书如果把习题都解答出来了，也就至少会降低一半价值．但是当我们仔细看了一下习题后，就发现，这里几乎没有作样划葫芦般的模仿性操作，也几乎没有什么技巧性的“难题”．书中的习

题主要是帮助读者领略或多少掌握一些现代数学的风格和表述方法．这就不应该是只靠大学生自己单枪匹马地探索解题方法．所以我们仍然选了一些有代表性的题目，阐述了我们自己的想法．但是这些题目也没有完全做到底，读者自己仍然要下苦工夫，甚至查阅一些参考书．可以说，本书的“附录”只是一个习题课的参考材料．书中仍有不少习题应该在参考书中去找解答．也因此，第 5 章后一部分习题就完全没有作提示了，因为那样会占用太多篇幅．同时原作者似乎也只是把重点放在前四章和第 5 章前一半．

不论如何，这个附录必有许多缺点：可能对习题理解有误，可能解法有误．至于有些题目做得“不好”是必然的，希望读者不吝赐教．但是如果这些提示对读者有所启发而引发动手解题的欲望，也就完全达到了目的．

原书提到了少量参考书，不仅如原书作者说的可能不完备，甚至对求解习题也不会是立竿见影的．我们也不打算再多列一些．至少，所有关于微分流形、微分拓扑的书，大部分是可以用的．但想要达到上面讲的目的，再读一本篇幅大一点的书似乎是不可少的．但译者想请大家注意原作者的一本大部头书（五卷集）：

M. Spivak. *A Comprehensive Introduction to Differential Geometry.* Publish or Perish, Inc. Berkeley, 1979.

这是一本几何名著，第一卷则是讲微分流形的．读者如果有可能下一点功夫至少读上前几章，必会大有收获——但也不一定能找到这里的习题的详细解答．

不少同志用过本书作为教材．浙江大学干丹岩教授把自己教学中发现的原书的错误特别详细告知，在修订译本时大都作了修正．武汉大学杜乃林教授在编写习题解答时给了我们极大的帮助．在此表示诚挚的谢意．这次出版译者自己也改正了一些错误．我们诚恳地欢迎读者继续提出批评意见．

译者

2005 年国庆节

原 书 序

“高等微积分”中有一些部分，由于其概念和方法比较复杂，所以在初等水平上难以严格处理．本书就是专门讲述这些部分的．这里采用的探讨方法是深奥数学中初等形式的现代方法．作为正式要求的预备知识只需要一学期的线性代数知识、对集合论的记号略有所知、以及一门内容得体的大学一年级微积分课［其中至少应提到实数集合的上确界（sup）与下确界（inf）］．除此之外，对抽象数学一定程度的熟悉（哪怕是潜在的）则几乎是不可缺少的．

本书前半部的内容是高等微积分中的简单部分，它把初等微积分中的一些内容推广到高维．第 1 章是预备知识，第 2 章、第 3 章讨论微分和积分．

本书其余部分用于研究曲线、曲面和更高维的类似物．这里，现代的和经典的处理方式按照完全不相同的思路进行．其间有许多交汇点，最终汇合在最后一节．本书封面上复印的那个很经典的方程也就是本书最后的一个定理．这个定理（斯托克斯定理）具有奇妙的历史，它已经历过惊人的变化．

这个定理的第一个提法出现在威廉·汤姆森爵士（Sir William Thomson）［即后来的开尔文勋爵（Lord Kelvin）］1850 年 7 月 2 日致斯托克斯的信末附笔中．它公开出现则是在 1854 年，作为当年史密斯奖竞赛的第 8 题．这个竞赛由斯托克斯教授主持，每年由剑桥大学最好的数学学生参加．到他去世之时，这个结果就广为人知了．人们将其命名为斯托克斯定理．他的同时代人至少对此给出过三个证明：汤姆森发表了第一个，另一个见于汤姆森和泰特所著《论自然哲学》（Thomson and Tait, *Treatise on Natural Philosophy*），麦克斯韦（Maxwell）在《电磁论》（*Electricity and Magnetism*）[13] 中又给出了一个证明．此后，斯托克斯的名字被用于广泛得多的结果，在数学的某些领域的发展中显然如此重要，以致斯托克斯定理可以看作

研究“推广”方法价值的一个例证.

本书中斯托克斯定理有三种形式. 斯托克斯本人得到的形式在最后一节，还有和它不可分离的伴随定理——格林（Green）定理和散度定理，这三个定理，也就是本书副标题里讲的经典定理，很容易从一个现代的斯托克斯定理推导出来，后者出现在第5章靠前部分. 经典定理关于曲线和曲面所讲的内容就是这个现代的斯托克斯定理对它们的高维类似物（流形）所谈的内容. 第5章第1节彻底地研究了流形. 研究流形的理由只能从它在现代数学中的重要性来说明，其实研究它并不比仅仅详细研究曲线曲面更花力气.

读者可能会以为现代斯托克斯定理至少和可以由它导出的经典定理一样难. 其实不然，它只不过是斯托克斯定理的另外一种讲法很简单的推论. 这个很抽象的讲法是第4章最后的也是主要的结果. 完全有理由设想，迄今回避了的难点必然隐藏在这里. 然而这个定理的证明，在数学家看来，却是自明的——只是直接的计算而已. 但另一方面，如果没有第4章一大堆艰难的定义，这个自明的陈述都无法理解. 这里有一些好的理由说明为什么定理如此容易而定义却很难. 斯托克斯定理的发展提示了，一个简单的原理可以化装成好几个艰深的结果. 许多定理的证明只不过是撕掉这层伪装罢了，另一方面，定义却提供了双重目的：它们既以严格的概念代替模糊的概念，又是非常好的证明工具. 第4章前两节确切地定义了经典数学中所谓“微分表达式” $Pdx + Qdy + Rdz$ 或 $Pdxdy + Qdydz + Rdzdx$ 是什么，并且证明了它们的运算规则. 第3节定义的链，以及单位分解（在第3章里已介绍)，使我们不必在证明中把流形切成小块. 它们把有关流形的问题化成关于欧几里得空间的问题. 在流形里每一件东西看来都很难，而在欧几里得空间里，每一件东西却都很容易.

把一个主题的深奥之处集中到定义上去，无可否认是很经济的，但这必定会对读者造成一些困难. 我希望读者鼓起勇气彻底学好第4章，确信花的工夫是值得的：最后一节的经典定理只是第4章的应用中少量的几个，而绝不是最重要的应用. 许多其他的应用放在习

题里，读者查一下参考文献还可以找到进一步的发展.

关于习题和参考文献还要讲几句，本书每节末都有习题，并且（和定理一样）按章编号. 加了星号的问题表明正文要用到其结果，但是这种谨慎应当是不必要的——习题是本书最重要的部分，读者至少应该对所有题目都试一试. 参考文献必然编得或者很不完备或者繁冗不堪，因为至少有一半主要的数学分支都可以很有根据地推荐为本书内容的合理的继续. 我试图把它编得虽不完备但却很诱人.

我借重印这本书的机会改正热情的读者们向我指出的许多印刷和原稿中的小错误. 此外，定理 3-11 以后的材料已完全修订和改正过了. 另一些重要的改变，如果放进正文中，势必作过大的改动，所以放在书末的补遗里.

Michael Spivak

1968 年 3 月　于马萨诸塞州 沃尔瑟姆市

Preface

This little book is especially concerned with those portions of "advanced calculus" in which the subtlety of the concepts and methods makes rigor difficult to attain at an elementary level. The approach taken here uses elementary versions of modern methods found in sophisticated mathematics. The formal prerequisites include only a term of linear algebra, a nodding acquaintance with the notation of set theory, and a respectable first-year calculus course (one which at least mentions the least upper bound (sup) and greatest lower bound (inf) of a set of real numbers). Beyond this a certain (perhaps latent) rapport with abstract mathematics will be found almost essential.

The first half of the book covers that simple part of advanced calculus which generalizes elementary calculus to higher dimensions. Chapter 1 contains preliminaries, and Chapters 2 and 3 treat differentiation and integration.

The remainder of the book is devoted to the study of curves, surfaces, and higher-dimensional analogues. Here the modern and classical treatments pursue quite different routes; there are, of course, many points of contact, and a significant encounter

occurs in the last section. The very classical equation reproduced on the cover appears also as the last theorem of the book. This theorem (Stokes' Theorem) has had a curious history and has undergone a striking metamorphosis.

The first statement of the Theorem appears as a postscript to a letter, dated July 2, 1850, from Sir William Thomson (Lord Kelvin) to Stokes. It appeared publicly as question 8 on the Smith's Prize Examination for 1854. This competitive examination, which was taken annually by the best mathematics students at Cambridge University, was set from 1849 to 1882 by Professor Stokes; by the time of his death the result was known universally as Stokes' Theorem. At least three proofs were given by his contemporaries: Thomson published one, another appeared in Thomson and Tait's *Treatise on Natural Philosophy*, and Maxwell provided another in *Electricity and Magnetism* [13]. Since this time the name of Stokes has been applied to much more general results, which have figured so prominently in the development of certain parts of mathematics that Stokes' Theorem may be considered a case study in the value of generalization.

In this book there are three forms of Stokes' Theorem. The version known to Stokes appears in the last section, along with its inseparable companions, Green's Theorem and the Divergence Theorem. These three theorems, the classical theorems of the subtitle, are derived quite easily from a modern Stokes' Theorem which appears earlier in Chapter 5. What the classical theorems state for curves and surfaces, this theorem states for the higher-dimensional analogues (manifolds) which are studied thoroughly in the first part of Chapter 5. This study of manifolds, which could be justified solely on the basis of their importance in modern mathematics, actually involves no more effort than a careful study of curves and surfaces alone would require.

The reader probably suspects that the modern Stokes' Theorem is at least as difficult as the classical theorems derived from it. On the contrary, it is a very simple consequence of yet another version of Stokes' Theorem; this very abstract version is the final and main result of Chapter 4.

It is entirely reasonable to suppose that the difficulties so far avoided must be hidden here. Yet the proof of this theorem is, in the mathematician's sense, an utter triviality—a straightforward computation. On the other hand, even the statement of this triviality cannot be understood without a horde of difficult definitions from Chapter 4. There are good reasons why the theorems should all be easy and the definitions hard. As the evolution of Stokes' Theorem revealed, a single simple principle can masquerade as several difficult results; the proofs of many theorems involve merely stripping away the disguise. The definitions, on the other hand, serve a twofold purpose: they are rigorous replacements for vague notions, and machinery for elegant proofs. The first two sections of Chapter 4 define precisely, and prove the rules for manipulating, what are classically described as "expressions of the form" $P\,dx + Q\,dy + R\,dz$, or $P\,dx\,dy + Q\,dy\,dz + R\,dz\,dx$. Chains, defined in the third section, and partitions of unity (already introduced in Chapter 3) free our proofs from the necessity of chopping manifolds up into small pieces; they reduce questions about manifolds, where everything seems hard, to questions about Euclidean space, where everything is easy.

Concentrating the depth of a subject in the definitions is undeniably economical, but it is bound to produce some difficulties for the student. I hope the reader will be encouraged to learn Chapter 4 thoroughly by the assurance that the results will justify the effort: the classical theorems of the last section represent only a few, and by no means the most important, applications of Chapter 4; many others appear as problems, and further developments will be found by exploring the bibliography.

The problems and the bibliography both deserve a few words. Problems appear after every section and are numbered (like the theorems) within chapters. I have starred those problems whose results are used in the text, but this precaution should be unnecessary—the problems are the most important part of the book, and the reader should at least attempt them all. It was necessary to make the bibliography either very incomplete or unwieldy, since half the major

branches of mathematics could legitimately be recommended as reasonable continuations of the material in the book. I have tried to make it incomplete but tempting.

Many criticisms and suggestions were offered during the writing of this book. I am particularly grateful to Richard Palais, Hugo Rossi, Robert Seeley, and Charles Stenard for their many helpful comments.

I have used this printing as an opportunity to correct many misprints and minor errors pointed out to me by indulgent readers. In addition, the material following Theorem 3-11 has been completely revised and corrected. Other important changes, which could not be incorporated in the text without excessive alteration, are listed in the Addenda at the end of the book.

Michael Spivak

Waltham, Massachusetts
March 1968

目　　录

Contents

第 1 章　欧几里得空间上的函数

1.1　范数与内积

欧几里得（Euclid）n 维空间（也简称欧氏空间）$\mathbf{R}^n$ 定义为一切实数 x^i 的 n 数组 $(x^1,\cdots,x^n)$（一个“1 数组”就是一个数，而 $\mathbf{R}^1=\mathbf{R}$ 则是一切实数的集）的集合．$\mathbf{R}^n$ 的元通常称为 $\mathbf{R}^n$ 的点，而 $\mathbf{R}^1$, $\mathbf{R}^2,\mathbf{R}^3$ 通常分别称为直线、平面和空间．如 x 表示 $\mathbf{R}^n$ 的一元素，则 x 是一个 n 数组，其中第 i 个记作 x^i；于是我们可以写成

$$x=(x^1,\cdots,x^n).$$

$\mathbf{R}^n$ 中的点也常常称为 $\mathbf{R}^n$ 中的向量，因为，按照 $x+y=(x^1+y^1,\cdots,x^n+y^n)$ 以及 $ax=(ax^1,\cdots,ax^n)$ 作为运算，$\mathbf{R}^n$ 是一个向量空间（在实数域上，维数为 n）．在这向量空间中，向量 x 的长度概念，通常称为 x 的范数 $|x|$，并定义为 $|x|=\sqrt{(x^1)^2+\cdots+(x^n)^2}$．如 $n=1$，则 $|x|$ 就是通常的 x 的绝对值．范数和 $\mathbf{R}^n$ 的向量空间结构间的如下关系极为重要．

定理 1-1　如 $x,y\in\mathbf{R}^n$ 且 $a\in\mathbf{R}$，则

(1) $|x|\geqslant 0$，当且仅当 $x=0$ 时 $|x|=0$.

(2) $\left|\sum_{i=1}^{n}x^iy^i\right|\leqslant|x|\cdot|y|$，当且仅当 x 与 y 线性相关时等式成立．

(3) $|x+y|\leqslant|x|+|y|$.

(4) $|ax|=|a|\cdot|x|$.

证

(1) 留给读者．

(2) 如 x 与 y 线性相关，等式明显成立．

如不是这样，则对一切 $\lambda \in \mathbf{R}$, $\lambda y - x \neq 0$, 因此

$$0 < |\lambda y - x|^2 = \sum_{i=1}^{n} (\lambda y^i - x^i)^2$$

$$= \lambda^2 \sum_{i=1}^{n} (y^i)^2 - 2\lambda \sum_{i=1}^{n} x^i y^i + \sum_{i=1}^{n} (x^i)^2.$$

所以右方是关于λ 的没有实根的二次式，其判别式必须为负．于是

$$4\left(\sum_{i=1}^{n} x^i y^i\right)^2 - 4\sum_{i=1}^{n} (x^i)^2 \cdot \sum_{i=1}^{n} (y^i)^2 < 0.$$

(3)
$$\begin{aligned} |x+y|^2 &= \sum_{i=1}^{n} (x^i + y^i)^2 \\ &= \sum_{i=1}^{n} (x^i)^2 + \sum_{i=1}^{n} (y^i)^2 + 2\sum_{i=1}^{n} x^i y^i \\ &\leqslant |x|^2 + |y|^2 + 2|x| \cdot |y| && \text{由(2)} \\ &= (|x| + |y|)^2. \end{aligned}$$

(4) $|ax| = \sqrt{\sum_{i=1}^{n} (ax^i)^2} = \sqrt{a^2 \sum_{i=1}^{n} (x^i)^2} = |a| \cdot |x|$. ▌

在(2)中出现的量 $\sum_{i=1}^{n} x^i y^i$ 称为 x 与 y 的内积并记作$\langle x,y \rangle$. 内积的一些最重要的性质如下所述．

定理 1-2　如 x, x_1, x_2 与 y, y_1, y_2 是 $\mathbf{R}^n$ 中的向量，且 $a \in \mathbf{R}$，则

(1) $\langle x,y \rangle = \langle y,x \rangle$ (对称性).

(2) $\langle ax,y \rangle = \langle x,ay \rangle = a\langle x,y \rangle$ (双线性)

$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$.

(3) $\langle x,x \rangle \geqslant 0$, 且$\langle x,x \rangle = 0$ 当且仅当 $x = 0$ (正定性).

(4) $|x| = \sqrt{\langle x,x \rangle}$.

(5) $\langle x,y \rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$ (极化等式).

证

(1) $\langle x,y\rangle = \sum_{i=1}^{n} x^i y^i = \sum_{i=1}^{n} y^i x^i = \langle y,x\rangle$.

(2) 由(1)只须证明

$$\langle ax,y\rangle = a\langle x,y\rangle,\langle x_1+x_2,y\rangle = \langle x_1,y\rangle + \langle x_2,y\rangle.$$

这些可由下列等式得出:

$$\langle ax,y\rangle = \sum_{i=1}^{n}(ax^i)y^i = a\sum_{i=1}^{n} x^i y^i = a\langle x,y\rangle,$$

$$\begin{aligned}\langle x_1+x_2,y\rangle &= \sum_{i=1}^{n}(x_1^i+x_2^i)y^i = \sum_{i=1}^{n} x_1^i y^i + \sum_{i=1}^{n} x_2^i y^i \\ &= \langle x_1,y\rangle + \langle x_2,y\rangle.\end{aligned}$$

(3)和(4)证明留给读者.

(5) $\dfrac{|x+y|^2-|x-y|^2}{4}$

$$= \frac{1}{4}[\langle x+y,x+y\rangle - \langle x-y,x-y\rangle] \qquad \text{由(4)}$$

$$= \frac{1}{4}[\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle - (\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle)]$$

$$= \langle x,y\rangle. \ \blacksquare$$

我们对记号作一些重要注解以结束本节. 向量 $(0,\cdots,0)$ 通常简记为**0**. $\mathbf{R}^n$ 的**通常基底**是 $e_1,\cdots,e_n$, 其中 $e_i = (0,\cdots,1,\cdots,0)$ 在第 i 个位置上是1. 如 $T:\mathbf{R}^n\to\mathbf{R}^m$ 是一个线性变换, T 关于 $\mathbf{R}^n$ 与 $\mathbf{R}^m$ 的通常基底的矩阵是 $m\times n$ 矩阵 $A=(a_{ij})$, 其中 $T(e_i) = \sum_{j=1}^{m} a_{ji}e_j$——$T(e_i)$ 的系数出现在矩阵的第 i 列. 如 $S:\mathbf{R}^m\to\mathbf{R}^p$ 有 $p\times m$ 矩阵 B , 则 $S\circ T$ 有 $p\times n$ 矩阵 BA [这里 $S\circ T(x) = S(T(x))$, 绝大多数线性代数书籍把 $S\circ T$ 简记为 ST], 为要找出 $T(x)$, 我们来计算 $m\times 1$ 矩阵

$$\begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \begin{pmatrix} a_{11},\cdots,a_{1n} \\ \vdots \qquad \vdots \\ a_{m1},\cdots,a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix};$$

则 $T(x) = (y^1, \cdots, y^m)$. 下一习惯记法大大简化许多公式：如 $x \in \mathbf{R}^n$ 与 $y \in \mathbf{R}^m$，则 (x,y) 表示

$$(x^1, \cdots, x^n, y^1, \cdots, y^m) \in \mathbf{R}^{n+m}.$$

习题

1-1. * 求证 $|x| \leqslant \sum_{i=1}^{n} |x^i|$.

1-2. 定理 1-1(3)中的等式何时成立？提示：重新检查证明；答案并不是"当 x 与 y 线性相关".

1-3. 求证 $|x - y| \leqslant |x| + |y|$. 何时等式成立？

1-4. 求证 $||x| - |y|| \leqslant |x - y|$.

1-5. 量 $|y - x|$ 称为 x 与 y 间的**距离**. 求证并在几何上解释"三角形不等式"：

$$|z - x| \leqslant |z - y| + |y - x|.$$

1-6. 设 f 与 g 在 $[a,b]$ 上平方可积，

(a) 求证 $\left|\int_a^b f \cdot g\right| \leqslant \left(\int_a^b f^2\right)^{1/2} \cdot \left(\int_a^b g^2\right)^{1/2}$ 提示：分别考虑以下两种情况：对某一 $\lambda \in \mathbf{R}$, $0 = \int_a^b (f - \lambda g)^2$；对一切 $\lambda \in \mathbf{R}$, $0 < \int_a^b (f - \lambda g)^2$.

(b) 如等式成立，$f = \lambda g$ 必定对某个 $\lambda \in \mathbf{R}$ 成立吗？如 f 与 g 连续又怎样？

(c) 证明定理 1-1(2)是(a)的一个极限情形.

1-7. 一线性变换 $T: \mathbf{R}^n \to \mathbf{R}^n$，如果 $|T(x)| = |x|$ 则称为**保范数的**，如果 $\langle Tx, Ty\rangle = \langle x, y\rangle$，则称为**保内积的**.

(a) 求证 T 是保范数的当且仅当 T 是保内积的.

(b) 求证这种线性变换 T 是 1-1 的，而且 T^{-1} 也是同一种变换.

1-8. 如 $x, y \in \mathbf{R}^n$ 不为零，x 与 y 间的(夹)角记作 $\angle(x,y)$ 定义为 $\arccos(\langle x, y\rangle / |x| \cdot |y|)$，由定理 1-1(2)这是有意义的. 线性变换 T 称为**保角的**，如 T 是 1-1 的，且对 $x, y \neq 0$，我们有 $\angle(Tx, Ty) = \angle(x,y)$.

(a) 求证：如 T 是保范数的，则 T 是保角的.

(b) 如 $\mathbf{R}^n$ 有一基底 $x_1, \cdots, x_n$，又有正数 $\lambda_1, \cdots, \lambda_n$ 使得 $Tx_i = \lambda_i x_i$，求证 T 是保角的当且仅当所有 λ_i 皆相等.[1]

---

1. 原书并不要求 λ_i 为正，而结论则是"当且仅当所有 $|\lambda_i|$ 相等"，是不正确的，因为有反例. 见"部分习题的解答或提示."——译者注

(c) 有哪些 $T:\mathbf{R}^n\to\mathbf{R}^n$ 是保角的?

1-9. 如 $0\leqslant\theta<\pi$,设 $T:\mathbf{R}^2\to\mathbf{R}^2$ 有矩阵$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. 求证 T 是保角的,且若 $x\neq 0$ 则 $\angle(x,Tx)=\theta$.

1-10. * 如 $T:\mathbf{R}^m\to\mathbf{R}^n$ 是一线性变换,证明有这样的数 M 使得对于 $h\in\mathbf{R}^m$ 有 $|T(h)|\leqslant M|h|$. 提示:用 $|h|$ 以及 T 的矩阵中的元估计 $|T(h)|$.

1-11. 如 $x,y\in\mathbf{R}^n$, z, $w\in\mathbf{R}^m$,证明 $\langle(x,z),(y,w)\rangle=\langle x,y\rangle+\langle z,w\rangle$ 以及 $|(x,z)|=\sqrt{|x|^2+|z|^2}$. 注意 (x,z) 与 (y,w) 表示 $\mathbf{R}^{n+m}$ 中的点.

1-12. * 设 $(\mathbf{R}^n)^*$ 表示向量空间 $\mathbf{R}^n$ 的对偶空间. 如 $x\in\mathbf{R}^n$,用 $\varphi_x(y)=\langle x,y\rangle$ 定义 $\varphi_x\in(\mathbf{R}^n)^*$. 用 $T(x)=\varphi_x$ 定义 $T:\mathbf{R}^n\to(\mathbf{R}^n)^*$. 证明 T 是一个1-1线性变换,并作出结论:每一个 $\varphi\in(\mathbf{R}^n)^*$ 是关于惟一的一个 $x\in\mathbf{R}^n$ 的 φ_x.

1-13. * 如 $x,y\in\mathbf{R}^n$,则若 $\langle x,y\rangle=0$ 就称 x 与 y **垂直**(或**正交**). 如 x 与 y 垂直,求证 $|x+y|^2=|x|^2+|y|^2$.

1.2 欧几里得空间的子集

闭区间 $[a,b]$ 在 $\mathbf{R}^2$ 中有一自然的类比. 这就是**闭矩形** $[a,b]\times[c,d]$,定义为一切数对 (x,y) 的全体,其中 $x\in[a,b]$, $y\in[c,d]$. 更一般地,如 $A\subset\mathbf{R}^m$, $B\subset\mathbf{R}^n$,则 $A\times B\subset\mathbf{R}^{m+n}$ 定义为一切 $(x,y)\in\mathbf{R}^{m+n}$ 的集,其中 $x\in A,y\in B$. 特别地, $\mathbf{R}^{m+n}=\mathbf{R}^m\times\mathbf{R}^n$. 如 $A\subset\mathbf{R}^m$, $B\subset\mathbf{R}^n$,和 $C\subset\mathbf{R}^p$,则 $(A\times B)\times C=A\times(B\times C)$,二者皆简记为 $A\times B\times C$. 这一记法也可推广到任意个数的集的乘积. 集 $[a_1,b_1]\times\cdots\times[a_n,b_n]\subset\mathbf{R}^n$ 称作 $\mathbf{R}^n$ 中的**闭矩形**,而集 $(a_1,b_1)\times\cdots\times(a_n,b_n)\subset\mathbf{R}^n$ 称作**开矩形**. 更一般地,一个集 $U\subset\mathbf{R}^n$ 称作**开集**(图1-1),如果对每一个 $x\in U$,有一个开矩形 A 使得 $x\in A\subset U$.

$\mathbf{R}^n$ 的一个子集 C 称为**闭集**如 $\mathbf{R}^n-C$ 是开集. 例如,如 C 只含有限多个点,则 C 是闭的. 读者应该补充证明: $\mathbf{R}^n$ 中的闭矩形确为一闭集.

如 $A\subset\mathbf{R}^n$ 且 $x\in\mathbf{R}^n$,则下列三种可能性之一必成立(图1-2).



图 1-1

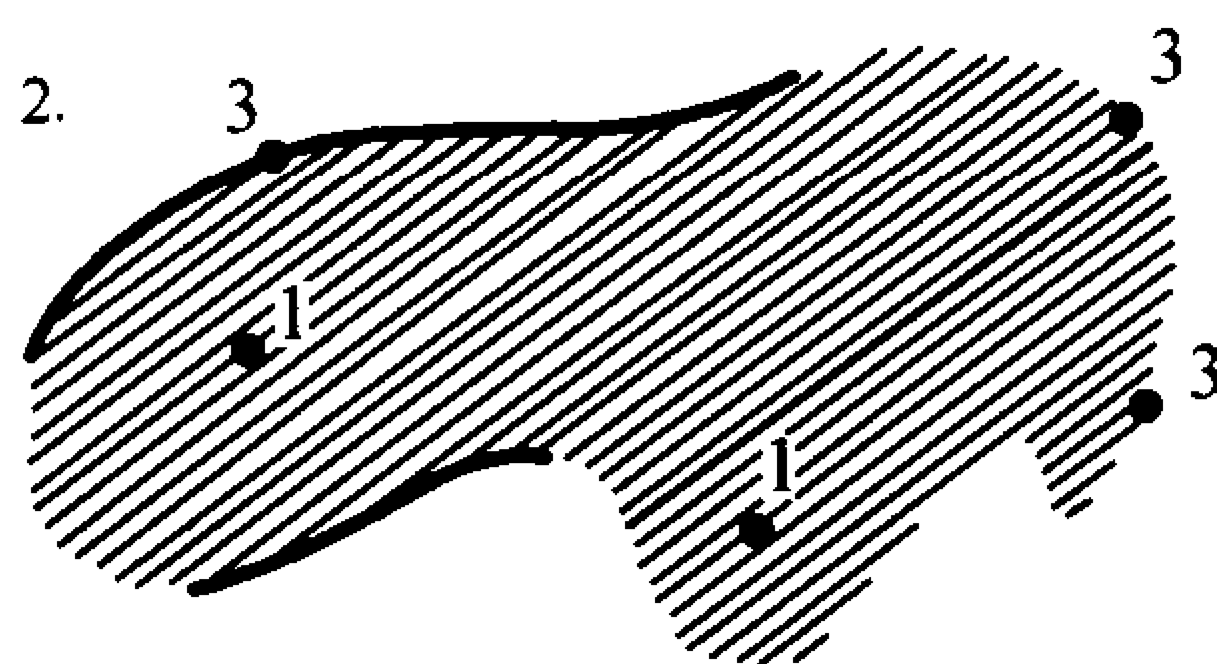


图 1-2

1. 存在一个开矩形 B 使得 $x \in B \subset A$.

2. 存在一个开矩形 B 使得 $x \in B \subset \mathbf{R}^n - A$.

3. 如 B 是任一个开矩形使 $x \in B$ 者，则 B 同时含有 A 与 $\mathbf{R}^n - A$ 的点.

满足(1)的那些点构成 A 的**内域**，满足(2)的那些点构成 A 的**外域**，满足(3)的那些点构成 A 的**边界**. 习题 1-16 到 1-18 表明这些术语有时可能有意想不到的意义.

不难看出，任何集 A 的内域是开的；对 A 的外域，它实际上是 $\mathbf{R}^n - A$ 的内域，所以也是如此. 于是(习题 1-14)它们的并集是开的，而所剩下的，即其边界，必定是闭的.

我们把一组开集称为A的一个**开覆盖**（或简称**覆盖**A）[1] $\mathcal{O}$，如果任一点$x \in A$是在$\mathcal{O}$的某开集中．例如，如$\mathcal{O}$是一切开区间$(a, a+1)$的集合，其中$a \in \mathbf{R}$，则$\mathcal{O}$是$\mathbf{R}$的一(开)覆盖．很明显，$\mathcal{O}$中的有限个开集不能覆盖$\mathbf{R}$，也不能覆盖$\mathbf{R}$的任一无界集．类似情况对有界集也可能发生．设对一切正整数$n>1$，$\mathcal{O}$是一切开区间$(1/n, 1-1/n)$的集合，则$\mathcal{O}$是$(0,1)$的一开覆盖，但$\mathcal{O}$中的有限个集仍不能覆盖$(0,1)$．虽然这一现象可能不会出现特别的坏处，但这种状况不会发生的集至关重要，它们已有一个特殊的名称：一集A称为**紧**的，如它的任何开覆盖$\mathcal{O}$存在一个有限个开集的族仍能覆盖A.

只有有限个点的集显然是紧的，包含0以及数$1/n$(对一切整数n)的无限集A也是紧的（理由：如$\mathcal{O}$是一覆盖，则存在$\mathcal{O}$中某一开集U有$0 \in U$，那么A中只有有限个别的点不在U中，每个这样的点至多只要再加一个开集就可以了）.

下列几个结果可大大简化对紧集的认识，其中只有第一个结果有一定的深度(也就是，用到了有关实数的一些性质).

定理 1-3（海涅-波雷耳（Heine Borel）） 闭区间$[a,b]$是紧的．

证 如$\mathcal{O}$是$[a,b]$的一个开覆盖，设$A=\{x : a \leqslant x \leqslant b$且$[a,x]$能被$\mathcal{O}$中某有限开集所覆盖$\}$.

注意$a \in A$，且A显然有上界（以b为上界)．我们希望证明$b \in A$. 这只是对$\alpha = A$的上确界证明两件事：(1) $\alpha \in A$，(2) $b=\alpha$就行了．

因$\mathcal{O}$是一覆盖，故对$\mathcal{O}$存在某一U有$\alpha \in U$. 那么在某区间中α左边的一切点也在U中(见图1-3)．因为α是A的上确界，故在这区间中有一$x \in A$．于是$[a,x]$能被$\mathcal{O}$中某有限个开集所覆盖，而$[x,\alpha]$被一个集U所覆盖．所以$[a,\alpha]$能被$\mathcal{O}$中有限个开集所覆盖，即$\alpha \in A$. 这就证明了(1).

要证(2)为真，假设不然：$\alpha<b$. 因此在α与b之间有一点x'使$[\alpha,x'] \subset U$．因$\alpha \in A$，区间$[a,\alpha]$能被$\mathcal{O}$中有限个开集所覆盖，而

1. 原文意思是若一个集族是A的覆盖，就说覆盖A. 而不是说开覆盖可以简称为覆盖，而应说开集族覆盖A. ——译者注

$[\alpha,x']$ 已被 U 覆盖．所以 $x'\in A$，这和 α 是 A 的上确界相矛盾．▌

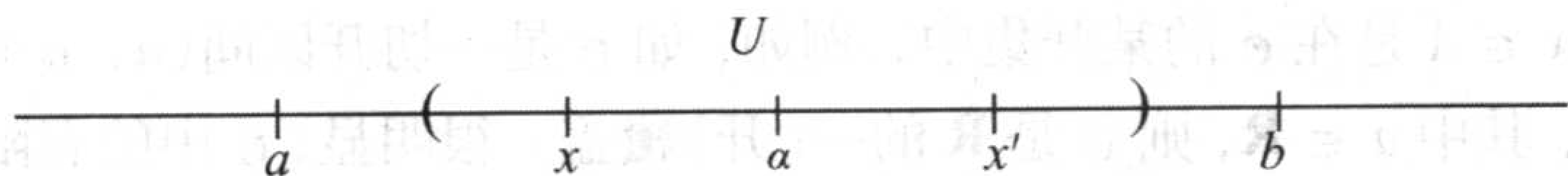


图 1-3

如 $B\subset \mathbf{R}^m$ 是紧的且 $x\in \mathbf{R}^n$，易见 $\{x\}\times B\subset \mathbf{R}^{n+m}$ 是紧的．但是，可以作出一个强得多的论断．

定理 1-4　如 B 是紧的，$\mathcal{O}$ 是 $\{x\}\times B$ 的一开覆盖，则有包含 x 的一开集 $U\subset \mathbf{R}^n$ 使得 $U\times B$ 能被 $\mathcal{O}$ 中有限个集所覆盖．

证　因为 $\{x\}\times B$ 是紧的，我们可以一开始就认为 $\mathcal{O}$ 是有限的，我们只要找出开集 U 使 $U\times B$ 能被 $\mathcal{O}$ 所覆盖．

对每一个 $y\in B$，点 (x,y) 在 $\mathcal{O}$ 的某开集 W 中．因 W 是开的，对某一开矩形 $U_y\times V_y$ 我们有 $(x,y)\in U_y\times V_y\subset W$．这些集 V_y 覆盖了紧集 B，所以有限个 $V_{y_1},\cdots,V_{y_k}$ 也覆盖 B. 令 $U=U_{y_1}\cap\cdots\cap U_{y_k}$．于是，如 $(x',y')\in U\times B$，对某一 i 我们有 $y'\in V_{y_i}$（图 1-4），当然 $x'\in U_{y_i}$．所以 $(x',y')\in U_{y_i}\times V_{y_i}$，它包含在 $\mathcal{O}$ 的某个 W 中．▌

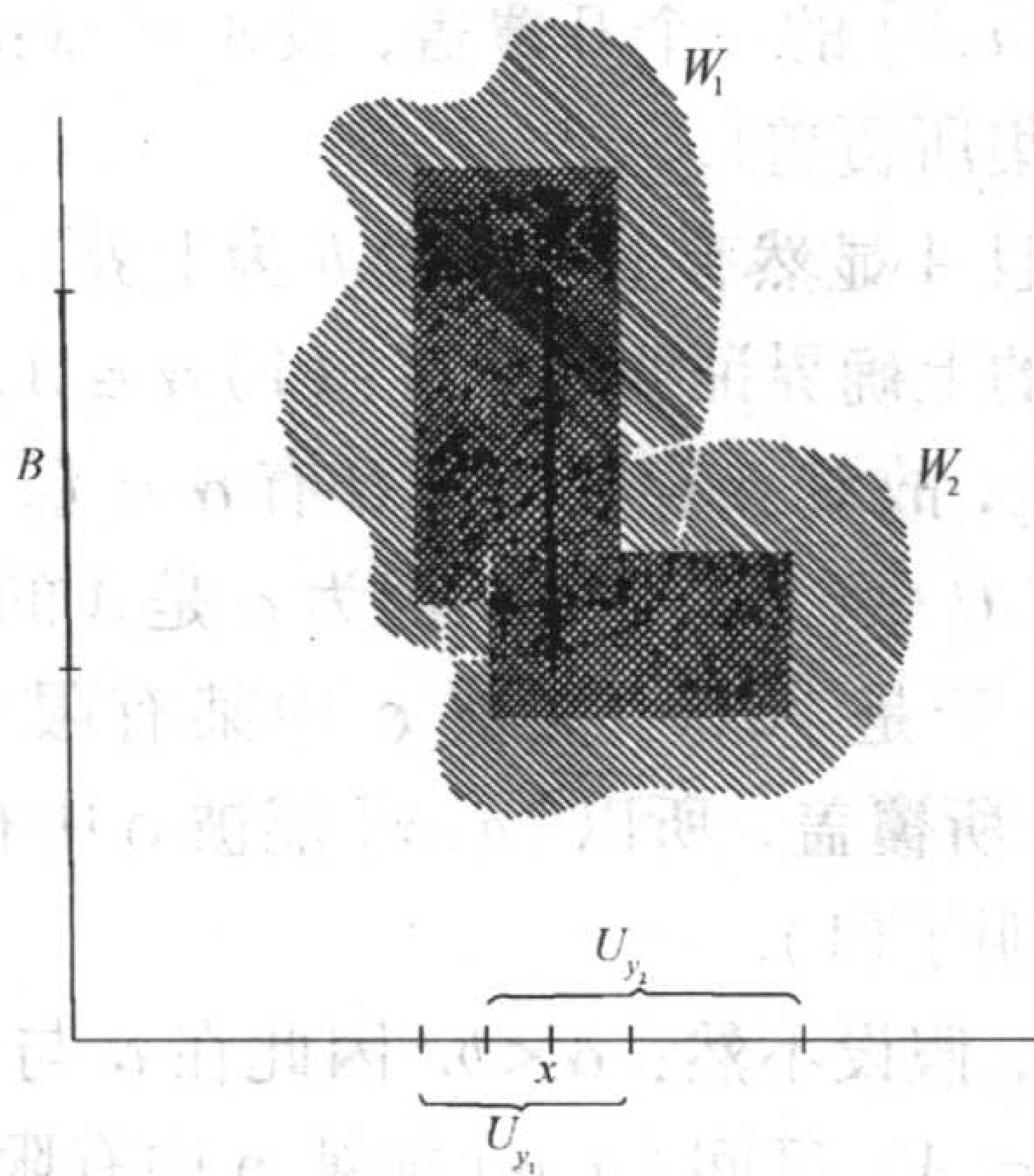


图 1-4

推论 1-5　如 $A\subset \mathbf{R}^n$ 与 $B\subset \mathbf{R}^m$ 是紧的，则 $A\times B\subset \mathbf{R}^{n+m}$ 也是紧的．

证　如 $\mathcal{O}$ 是 $A\times B$ 的一开覆盖，则对每一个 $x\in A$，$\mathcal{O}$ 覆盖了 $\{x\}\times$

B. 由定理1-4, 有一个包含 x 的开集 U_x, 使得 $U_x \times B$ 能被 $\mathcal{O}$ 中有限个集覆盖. 因为A 是紧的, U_x 中的有限个 $U_{x_1},\cdots,U_{x_n}$ 覆盖A. 因为 $\mathcal{O}$ 中有限个集覆盖每一个 $U_{x_i} \times B$, 所以 $\mathcal{O}$ 中有限个集也就整个覆盖了 $A \times B$. ▌

推论 1-6 如每一个 A_i 是紧的, 则 $A_1 \times \cdots \times A_k$ 也是紧的. 特别地, $\mathbf{R}^k$ 中的闭矩形是紧的.

推论 1-7 $\mathbf{R}^n$ 中的有界闭集是紧的.

(逆定理也真(习题 1-20).)

证 如 $A \subset \mathbf{R}^n$ 是有界闭的, 则对某一个闭矩形 B, $A \subset B$. 如 $\mathcal{O}$ 是 A 的一个开覆盖, 则 $\mathcal{O}$ 与 $\mathbf{R}^n - A$ 一起是 B 的一个开覆盖. 所以 $\mathcal{O}$ 中有限个集 $U_1,\cdots,U_n$, 可能再加上 $\mathbf{R}^n - A$, 覆盖了 B. 因此, $U_1,\cdots,U_n$ 覆盖了 A. ▌

习题

1-14.* 求证任何一个(即使是无穷多个)开集的并集是开的. 求证两个(从而有限个)开集的交集是开的. 给出对于无穷多个开集的一个反例.

1-15. 求证 $\{x \in \mathbf{R}^n : |x - a| < r\}$ 是开的(参见习题 1-27).

1-16. 求下列集的内域、外域和边界:

$$\{x \in \mathbf{R}^n : |x| \leqslant 1\}$$

$$\{x \in \mathbf{R}^n : |x| = 1\}$$

$$\{x \in \mathbf{R}^n : \text{每一 } x^i \text{ 是有理数}\}.$$

1-17. 构造一个集 $A \subset [0,1] \times [0,1]$, 使得 A 在每一条水平线和垂直线上至多只含一点, 但 A 的边界 $= [0,1] \times [0,1]$. 提示: 只要能保证 A 在正方形 $[0,1] \times [0,1]$ 的每 1/4 中含有点, 又在每 1/16 中含有点, 如此等等, 这就够了.

1-18. 如 $A \subset [0,1]$ 是这样一些开区间 (a_i, b_i) 的并集, 使得 $(0,1)$ 中的每一有理数包含在某个 (a_i, b_i) 内, 求证 A 的边界 $= [0,1] - A$.

1-19.* 如 A 是包含任何有理数 $r \in [0,1]$ 的一个闭集, 求证 $[0,1] \subset A$.

1-20. 求证推论 1-7 的逆: $\mathbf{R}^n$ 的紧集是闭有界集(参见习题 1-28).

1-21.* (a) 如 A 是闭的且 $x \notin A$, 求证存在一数 $d > 0$ 使对一切 $y \in A$ 有 $|y - x| \geqslant d$.

(b) 如 A 是闭的, B 是紧的, 且 $A \cap B = \phi$, 求证存在 $d>0$ 使对一切 $y \in A$

与 $x \in B$ 有 $|y-x| \geqslant d$. 提示：对每一个 $b \in B$ 找出包含 b 的一开集 U 使得这一关系式对 $x \in U \cap B$ 成立.

(c) 若 A 与 B 都是闭的但都不是紧的，试在 $\mathbf{R}^2$ 中给出一个反例.

1-22.* 如 U 是开的且 $C \subset U$ 是紧的，证明存在一紧集 D 使得 $C \subset$ D 的内域且 D $\subset U$.

1.3　函数与连续性

从 $\mathbf{R}^n$ 到 $\mathbf{R}^m$ 的一个**函数**（有时称为 n 个变元的（向量值）函数）是一个规则，它把 $\mathbf{R}^n$ 中的每一点对应到 $\mathbf{R}^m$ 中的某一点. 一个函数 f 使 x 所对应的点记作 $f(x)$. $f:\mathbf{R}^n \to \mathbf{R}^m$（按上、下文读作"$f$ 把 $\mathbf{R}^n$ 映入 $\mathbf{R}^m$"或"f 映 $\mathbf{R}^n$ 入 $\mathbf{R}^m$"）表明 $f(x) \in \mathbf{R}^m$ 是对 $x \in \mathbf{R}^n$ 定义的. 记号 $f:A \to \mathbf{R}^m$ 表示 $f(x)$ 仅对集 A 中的 x 有定义，A 称为 f 的**定义域**. 如 $B \subset A$，我们把 $f(B)$ 定义为对 $x \in B$ 的一切 $f(x)$ 的集，又若 $C \subset \mathbf{R}^m$，我们定义 $f^{-1}(C) = \{x \in A : f(x) \in C\}$. 记号 $f:A \to B$ 表示 $f(A) \subset B$.

通过作出一函数 $f:\mathbf{R}^2 \to \mathbf{R}$ 的图，我们可以得到它的一方便的表示，这个图就是一切形如 $(x, y, f(x, y))$ 的 3 数组的集，它实际上是 3 维空间中的一个图形(例如，见第 2 章图 2-1 和图 2-2).

若 $f, g:\mathbf{R}^n \to \mathbf{R}$，则函数 $f+g, f-g, f\cdot g$ 与 f/g 可以确切地像单变量情况一样来定义. 如 $f:A \to \mathbf{R}^m$, $g:B \to \mathbf{R}^p$，其中 $B \subset \mathbf{R}^m$，则**复合函数** $g \circ f$ 定义为 $g \circ f(x) = g(f(x))$；$g \circ f$ 的定义域是 $A \cap f^{-1}(B)$. 如 $f: A \to \mathbf{R}^m$ 是 1-1 的，也就是，当 $x \neq y$ 时，$f(x) \neq f(y)$，我们定义 $f^{-1}: f(A) \to \mathbf{R}^n$，这里要求 $f^{-1}(z)$ 是惟一的 $x \in A$ 并且 $f(x) = z$.

一个函数 $f:A \to \mathbf{R}^m$ 用 $f(x) = (f^1(x), \cdots, f^m(x))$ 确定 m 个**分量函数** $f^1, \cdots, f^m: A \to \mathbf{R}$，反过来，如果已给 m 个函数 $g_1, \cdots, g_m: A \to \mathbf{R}$，则有惟一的函数 $f: A \to \mathbf{R}^m$ 使得 $f^i = g_i$，即 $f(x) = (g_1(x), \cdots, g_m(x))$. 这个函数 f 将记作 $(g_1, \cdots, g_m)$，所以我们总有 $f = (f^1, \cdots, f^m)$. 如 $\pi: \mathbf{R}^n \to \mathbf{R}^n$ 是恒等函数，$\pi(x) = x$，则 $\pi^i(x) = x^i$；函数 π^i 称作第 i 个**投影函数**.

和单变量情况一样，记号 $\lim_{x\to a}f(x) = b$ 表示，当选取 x 足够接近于 a 但不等于 a 时，我们可以使 $f(x)$ 任意地接近于 b. 用数学术语讲，这表明：对任一数 $\varepsilon > 0$，存在一数 $\delta > 0$ 使对 f 的定义域中的一切满足 $0 < |x-a| < \delta$ 的 x 有 $|f(x)-b| < \varepsilon$. 函数 $f:A\to\mathbf{R}^m$ 称为在 $a\in A$ 连续，如果 $\lim_{x\to a}f(x) = f(a)$. $f:A\to\mathbf{R}^m$ 在每一 $a\in A$ 处连续就简称 f 是连续的. 关于连续性概念的有趣的意想不到的一点是，它可以不用极限来定义. 由下一定理得知，$f:\mathbf{R}^n\to\mathbf{R}^m$ 连续，当且仅当只要 $U\subset\mathbf{R}^m$ 是开的 $f^{-1}(U)$ 就是开的. 如 f 的定义域不是 $\mathbf{R}^n$ 的全部，则需要一稍微复杂的条件.

定理 1-8 如 $A\subset\mathbf{R}^n$，函数 $f: A\to\mathbf{R}^m$ 连续当且仅当对任一开集 $U\subset\mathbf{R}^m$ 存在某开集 $V\subset\mathbf{R}^n$ 使得 $f^{-1}(U) = V\cap A$.

证 设 f 连续. 如 $a\in f^{-1}(U)$，则 $f(a)\in U$. 因 U 是开的，故有开矩形 B 使 $f(a)\in B\subset U$. 因 f 在 a 点连续，我们只要把 x 选取在包含 a 的某充分小的矩形 C 内，就能保证 $f(x)\in B$. 对每一 $a\in f^{-1}(U)$ 这样做，并令 V 为所有这些 C 的并集. 显然 $f^{-1}(U) = V\cap A$. 其逆也类似，留给读者去证明. ▌

定理 1-8 的下一推断极为重要.

定理 1-9 如 $f:A\to\mathbf{R}^m$ 是连续的，其中 $A\subset\mathbf{R}^n$，而 A 是紧的，则 $f(A)\subset\mathbf{R}^m$ 也是紧的.

证 设 $\mathcal{O}$ 是 $f(A)$ 的一个开覆盖. 对于 $\mathcal{O}$ 中每一个开集 U 存在一个开集 V_U 使得 $f^{-1}(U) = V_U\cap A$. 一切 V_U 的集合是 A 的一开覆盖. 因 A 是紧的，故有有限个 $V_{U_1},\cdots,V_{U_n}$ 覆盖 A. 于是 $U_1,\cdots,U_n$ 覆盖 $f(A)$. ▌

若 $f:A\to\mathbf{R}$ 有界，则 f 在 $a\in A$ 处不连续的程度可以用一个确切的方法加以度量. 对 $\delta > 0$，令

$$M(a,f,\delta) = \sup\{f(x): x\in A \text{ 且 } |x-a| < \delta\},$$

$$m(a,f,\delta) = \inf\{f(x): x\in A \text{ 且 } |x-a| < \delta\}.$$

f 在 a 处的**振幅** $o(f,a)$ 定义为 $o(f,a) = \lim_{\delta\to 0}[M(a,f,\delta) - m(a,f,\delta)]$. 因为 $M(a,f,\delta) - m(a,f,\delta)$ 当 δ 下降时也下降，所以这一极限恒存在. 关于 $o(f,a)$ 有两个重要事实.

定理 1-10　有界函数 f 当且仅当 $o(f,a)=0$ 时在 a 点连续.

证　设 f 在 a 点连续. 对任一数 $\varepsilon>0$ 我们可以选取一数 $\delta>0$ 使对一切 $x\in A$ 且 $|x-a|<\delta$ 者恒有 $|f(x)-f(a)|<\varepsilon$. 于是 $M(a,f,\delta)-m(a,f,\delta)\leqslant 2\varepsilon$. 因这对任何 ε 为真, 故有 $o(f,a)=0$. 其逆证法类似, 并留给读者. ▌

定理 1-11　设 $A\subset\mathbf{R}^n$ 是闭的. 如 $f:A\to\mathbf{R}$ 是任一有界函数, 又 $\varepsilon>0$, 则 $\{x\in A:o(f,x)\geqslant\varepsilon\}$ 是闭的.

证　设 $B=\{x\in A:o(f,x)\geqslant\varepsilon\}$. 我们要证明 $\mathbf{R}^n-B$ 是开的. 如果 $x\in\mathbf{R}^n-B$, 那么或者有 $x\notin A$, 不然的话, 就有 $x\in A$ 以及 $o(f,x)<\varepsilon$. 在第一种情况下, 因 A 是闭的, 故存在包含 x 的开矩形 C 使得 $C\subset\mathbf{R}^n-A\subset\mathbf{R}^n-B$. 在第二种情况下, 存在一 $\delta>0$ 使得 $M(x,f,\delta)-m(x,f,\delta)<\varepsilon$. 令 C 是一包含 x 的开矩形, 使对一切 $y\in C$, 有 $|x-y|<\delta$. 则若 $y\in C$, 就有一 δ_1, 使对所有满足 $|z-y|<\delta_1$ 的 z 有 $|x-z|<\delta$. 于是 $M(y,f,\delta_1)-m(y,f,\delta_1)<\varepsilon$, 从而 $o(y,f)<\varepsilon$. 所以 $C\subset\mathbf{R}^n-B$. ▌

习题

1-23. 若 $f:A\to\mathbf{R}^m$ 且 $a\in A$, 证明 $\lim_{x\to a}f(x)=b$ 当且仅当对于 $i=1,\cdots,m$ 有 $\lim_{x\to a}f^i(x)=b^i$.

1-24. 求证 $f:A\to\mathbf{R}^m$ 在 a 点连续当且仅当每一个 f^i 都如此.

1-25. 求证线性变换 $T:\mathbf{R}^n\to\mathbf{R}^m$ 是连续的. 提示: 利用习题 1-10.

1-26. 设 $A=\{(x,y)\in\mathbf{R}^2:x>0$ 且 $0<y<x^2\}$.

(a) 证明通过 $(0,0)$ 的任一直线包含一个以 $(0,0)$ 为内点的在 $\mathbf{R}^2-A$ 中的区间.

(b) 这样定义 $f:\mathbf{R}^2\to\mathbf{R}$, 当 $x\notin A$ 时 $f(x)=0$, 当 $x\in A$ 时 $f(x)=1$. 对 $h\in\mathbf{R}^2$ 定义 $g_h:\mathbf{R}\to\mathbf{R}$, $g_h(t)=f(th)$. 求证每个 g_h 在 0 点连续, 但 f 在 $(0,0)$ 点不连续.

1-27. 由考察 $f(x)=|x-a|$ 确定的 $f:\mathbf{R}^n\to\mathbf{R}$ 来证明 $\{x\in\mathbf{R}^n:|x-a|<r\}$ 是开的.

1-28. 如 $A\subset\mathbf{R}^n$ 不是闭的, 证明存在一无界的连续函数 $f:A\to\mathbf{R}$. 提示: 如

$x \in \mathbf{R}^n - A$ 但 $x \notin$ [$(\mathbf{R}^n - A)$ 的内域]，令 $f(y) = 1/|y - x|$.

1-29. 如 A 是紧的，求证任何连续函数 $f: A \to \mathbf{R}$ 有最大值和最小值.

1-30. 设 $f: [a,b] \to \mathbf{R}$ 是一增函数. 如 $x_1, \cdots, x_n \in [a,b]$ 各数不同，证明

$$\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a).$$

第2章 微　　分

2.1 基本定义

回想一函数$f:\mathbf{R}\to\mathbf{R}$在$a\in\mathbf{R}$处可微是指：存在$f'(a)$使得

$$(1)\qquad \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a).$$

对一般情形的函数$f:\mathbf{R}^n\to\mathbf{R}^m$，这个式子当然没有意义，但可以用一种方式将其重写使之有意义．如$\lambda:\mathbf{R}\to\mathbf{R}$是由$\lambda(h)=f'(a)\cdot h$定义的线性变换，则(1)式等价于

$$(2)\qquad \lim_{h\to 0}\frac{f(a+h)-f(a)-\lambda(h)}{h}=0.$$

(2)式常常可解释为$\lambda+f(a)$是f在a处的一个好的近似(见习题2-9)．因而我们集中注意力于线性变换λ，而把可微性定义重述如下：

函数$f:\mathbf{R}\to\mathbf{R}$在$a\in\mathbf{R}$点可微，如果有一线性变换$\lambda:\mathbf{R}\to\mathbf{R}$使得

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-\lambda(h)}{h}=0.$$

在这一形式下，这个定义对于高维有简单的推广：

函数$f:\mathbf{R}^n\to\mathbf{R}^m$在$a\in\mathbf{R}^n$点可微，如果存在一线性变换$\lambda:\mathbf{R}^n\to\mathbf{R}^m$使得

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-\lambda(h)|}{|h|}=0.$$

注意h是$\mathbf{R}^n$中的点，$f(a+h)-f(a)-\lambda(h)$是$\mathbf{R}^m$中的点，所以范

数记号是不可少的．这个线性变换λ 记作 $Df(a)$，称作f在a点的**导数**．短语"这个线性变换λ"的正确性证明如下．

定理 2-1 如$f:\mathbf{R}^n\to\mathbf{R}^m$ 在 $a\in\mathbf{R}^n$ 点可微，则存在一个惟一的线性变换 $\lambda:\mathbf{R}^n\to\mathbf{R}^m$ 使得

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-\lambda(h)|}{|h|}=0.$$

证 假定$\mu:\mathbf{R}^n\to\mathbf{R}^m$ 也满足

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-\mu(h)|}{|h|}=0.$$

令 $d(h)=f(a+h)-f(a)$，则

$$\begin{aligned}&\lim_{h\to 0}\frac{|\lambda(h)-\mu(h)|}{|h|}\\ &=\lim_{h\to 0}\frac{|\lambda(h)-d(h)+d(h)-\mu(h)|}{|h|}\\ &\leqslant\lim_{h\to 0}\frac{|\lambda(h)-d(h)|}{|h|}+\lim_{h\to 0}\frac{|d(h)-\mu(h)|}{|h|}=0.\end{aligned}$$

另一方面，因 $\frac{|\lambda(h)-\mu(h)|}{|h|}\geqslant 0$，所以 $\lim_{h\to 0}\frac{|\lambda(h)-\mu(h)|}{|h|}=0$. 如 $x\in\mathbf{R}^n$，则当 $t\to 0$ 时 $tx\to 0$. 因此对 $x\neq 0$ 我们有

$$0=\lim_{t\to 0}\frac{|\lambda(tx)-\mu(tx)|}{|tx|}=\frac{|\lambda(x)-\mu(x)|}{|x|}.$$

所以 $\lambda(x)=\mu(x)$. ▌

我们以后将会发现求 $Df(a)$ 的一个简单方法．目前我们来考察由 $f(x,y)=\sin x$ 定义的函数 $f:\mathbf{R}^2\to\mathbf{R}$. 那么 $Df(a,b)=\lambda$ 满足 $\lambda(x,y)=(\cos a)\cdot x$. 为要证明它，注意

$$\begin{aligned}&\lim_{(h,k)\to 0}\frac{|f(a+h,b+k)-f(a,b)-\lambda(h,k)|}{|(h,k)|}\\ &=\lim_{(h,k)\to 0}\frac{|\sin(a+h)-\sin a-(\cos a)\cdot h|}{|(h,k)|}.\end{aligned}$$

因为 $\sin'(a) = \cos a$，我们有

$$\lim_{h\to 0}\frac{|\sin(a+h)-\sin a-(\cos a)\cdot h|}{|h|}=0.$$

因为 $|(h,k)|\geqslant|h|$，所以还有：

$$\lim_{h\to 0}\frac{|\sin(a+h)-\sin a-(\cos a)\cdot h|}{|(h,k)|}=0.$$

考察 $Df(a):\mathbf{R}^n\to\mathbf{R}^m$ 关于 $\mathbf{R}^n$ 与 $\mathbf{R}^m$ 的通常基底的矩阵，常常是方便的．这个 $m\times n$ 矩阵称为 f 在 a 处的**雅可比**（Jacobi）**矩阵**，记作 $f'(a)$．如 $f(x,y)=\sin x$，则 $f'(a,b)=(\cos a,0)$．如 f: $\mathbf{R}\to\mathbf{R}$，则 $f'(a)$ 是 1×1 矩阵，其惟一元就是在初等微积分中记作 $f'(a)$ 的那个数．

如果 f 仅在包含 a 的某个开集上定义，那么还可以定义 $Df(a)$．为使定理的叙述流畅而又不失其普遍性，我们只考虑定义在 $\mathbf{R}^n$ 上的函数．设有 $f:\mathbf{R}^n\to\mathbf{R}^m$，如果 f 在每一个 $a\in A$ 处可微就称 f 在 A 上可微．如 $f:A\to\mathbf{R}^m$，又若 f 可以扩张为在包含 A 的某开集上的可微函数，则称 f 是可微的．

习题

2-1.* 求证：如 $f:\mathbf{R}^n\to\mathbf{R}^m$ 在 $a\in\mathbf{R}^n$ 处可微，则它在 a 点连续．提示：利用习题 1-10.

2-2. 一函数 $f:\mathbf{R}^2\to\mathbf{R}$，如对每一 $x\in\mathbf{R}$，对所有 $y_1,y_2\in\mathbf{R}$ 我们都有 $f(x,y_1)=f(x,y_2)$，则称 f **与第二变元无关**．试证 f 与第二变元无关当且仅当存在一函数 $g:\mathbf{R}\to\mathbf{R}$ 使得 $f(x,y)=g(x)$．$f'(a,b)$ 用 g' 表示时是什么？

2-3. 试决定何时一函数 f: $\mathbf{R}^2\to\mathbf{R}$ 与第一变元无关，并对这种 f 求出 $f'(a,b)$．什么样的函数既与第一变元无关也与第二变元无关？

2-4. 设 g 是单位圆周 $\{x\in\mathbf{R}^2:|x|=1\}$ 上的连续函数且有 $g(0,1)=g(1,0)=0$，$g(-x)=-g(x)$．定义 $f:\mathbf{R}^2\to\mathbf{R}$ 为：

$$f(x)=\begin{cases}|x|\cdot g\left(\dfrac{x}{|x|}\right) & x\neq 0,\\ 0 & x=0.\end{cases}$$

(a) 如 $x\in\mathbf{R}^2$ 且 $h:\mathbf{R}\to\mathbf{R}$ 定义为 $h(t)=f(tx)$，证明 h 是可微的．

(b) 证明f在$(0,0)$处不可微，除非$g=0$. 提示：当$k=0$时（再当$h=0$时）考察(h,k)，先证明$Df(0,0)$必须是零.

2-5. 设$f:\mathbf{R}^2\to\mathbf{R}$用下式定义：

$$f(x,y)=\begin{cases}\dfrac{x\,|y|}{\sqrt{x^2+y^2}} & (x,y)\neq 0,\\ 0 & (x,y)=0.\end{cases}$$

证明f是习题2-4中考虑过的那种函数，所以f在$(0,0)$处不可微.

2-6. 设$f:\mathbf{R}^2\to\mathbf{R}$定义为$f(x,y)=\sqrt{|xy|}$. 求证$f$在$(0,0)$处不可微.

2-7. 设$f:\mathbf{R}^n\to\mathbf{R}$是一函数使得$|f(x)|\leqslant|x|^2$. 证明$f$在0处可微.

2-8. 设$f:\mathbf{R}\to\mathbf{R}^2$. 求证：当且仅当$f^1$与$f^2$在$a\in\mathbf{R}$处可微时，$f$在$a$处可微，且这时

$$f'(a)=\begin{pmatrix}(f^1)'(a)\\(f^2)'(a)\end{pmatrix}.$$

2-9. 两函数$f,g:\mathbf{R}\to\mathbf{R}$称为**在$a$点直到$n$阶相等**，如果

$$\lim_{h\to 0}\frac{f(a+h)-g(a+h)}{h^n}=0.$$

(a) 试证：f在a点可微，当且仅当f在a点连续，且存在形如$g(x)=a_0+a_1(x-a)$的函数g使得f与g在a点直到一阶相等.

(b) 如$f'(x),\cdots,f^{(n)}(x)$在$x=a$附近存在，$f^{(n)}(x)$在a点连续，试证f与下式

$$g(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$$

定义的函数g在a点直到n阶相等. 提示：极限

$$\lim_{x\to a}\frac{f(x)-\displaystyle\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n}$$

可用罗必达（L'Hospital）法则计算.

2.2 基本定理

定理2-2(锁链规则)　如$f:\mathbf{R}^n\to\mathbf{R}^m$在$a$点可微，$g:\mathbf{R}^m\to\mathbf{R}^p$在$f(a)$

点可微，则其复合 $g\circ f:\mathbf{R}^n\to\mathbf{R}^p$ 在 a 点可微，且

$$D(g\circ f)(a) = Dg(f(a))\circ Df(a).$$

注．此式可写成

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a).$$

如 $m=n=p=1$，我们便得到老的锁链规则．

证 令 $b=f(a)$，$\lambda = Df(a)$，$\mu = Dg(f(a))$．如果我们定义

(1) $\varphi(x) = f(x) - f(a) - \lambda(x-a)$,

(2) $\psi(y) = g(y) - g(b) - \mu(y-b)$,

(3) $\rho(x) = g\circ f(x) - g\circ f(a) - \mu\circ\lambda(x-a)$,

则

(4) $\lim\limits_{x\to a}\dfrac{|\varphi(x)|}{|x-a|} = 0$,

(5) $\lim\limits_{y\to b}\dfrac{|\psi(y)|}{|y-b|} = 0$,

而我们必须证明

$$\lim_{x\to a}\frac{|\rho(x)|}{|x-a|} = 0.$$

现在

$$\begin{aligned}\rho(x) &= g(f(x)) - g(b) - \mu(\lambda(x-a))\\ &= g(f(x)) - g(b) - \mu(f(x)-f(a)-\varphi(x)) \quad \text{由(1)}\\ &= [g(f(x)) - g(b) - \mu(f(x)-f(a))] + \mu(\varphi(x))\\ &= \psi(f(x)) + \mu(\varphi(x)) \quad \text{由(2)}.\end{aligned}$$

于是我们必须证明

(6) $\lim\limits_{x\to a}\dfrac{|\psi(f(x))|}{|x-a|} = 0$,

(7) $\lim\limits_{x\to a}\dfrac{|\mu(\varphi(x))|}{|x-a|} = 0.$

(7)式容易从(4)式和习题1-10推得．如果$\varepsilon > 0$，从(5)式推知，对某一个$\delta > 0$我们有

$$|\psi(f(x))| < \varepsilon|f(x) - b|，只要|f(x) - b| < \delta,$$

而这一点只要对某个δ_1，由$|x - a| < \delta_1$就总成立．于是，由习题1-10，对某个M，

$$\begin{aligned}|\psi(f(x))| &< \varepsilon|f(x) - b| \\ &= \varepsilon|\varphi(x) + \lambda(x - a)| \\ &\leqslant \varepsilon|\varphi(x)| + \varepsilon M|x - a|.\end{aligned}$$

(6)式现就容易得出．▌

定理 2-3

(1) 如$f:\mathbf{R}^n \to \mathbf{R}^m$是一常值函数（也就是，若对某$y \in \mathbf{R}^m$，我们有：对一切$x \in \mathbf{R}^n$，$f(x) = y$），那么，

$$Df(a) = 0.$$

(2) 如$f:\mathbf{R}^n \to \mathbf{R}^m$是一个线性变换，则

$$Df(a) = f.$$

(3) 如$f:\mathbf{R}^n \to \mathbf{R}^m$，则$f$在$a \in \mathbf{R}^n$处可微当且仅当每个$f^i$是如此，且

$$Df(a) = (Df^1(a), \cdots, Df^m(a)).$$

于是$f'(a)$是$m \times n$矩阵，其第i行是$(f^i)'(a)$.

(4) 如$s:\mathbf{R}^2 \to \mathbf{R}$定义为$s(x,y) = x + y$，则

$$Ds(a,b) = s.$$

(5) 如$p:\mathbf{R}^2 \to \mathbf{R}$定义为$p(x,y) = x \cdot y$，则

$$Dp(a,b)(x,y) = bx + ay.$$

于是$p'(a,b) = (b,a)$.

证

(1) $\lim\limits_{h\to 0}\dfrac{|f(a+h)-f(a)-0|}{|h|}=\lim\limits_{h\to 0}\dfrac{|y-y-0|}{|h|}=0.$

(2) $\lim\limits_{h\to 0}\dfrac{|f(a+h)-f(a)-f(h)|}{|h|}$

$=\lim\limits_{h\to 0}\dfrac{|f(a)+f(h)-f(a)-f(h)|}{|h|}=0.$

(3) 如每一个f^i在a处可微，且

$$\lambda=(Df^1(a),\cdots,Df^m(a)),$$

则

$$\begin{aligned}&f(a+h)-f(a)-\lambda(h)\\&=(f^1(a+h)-f^1(a)-Df^1(a)(h),\cdots,\\&\quad f^m(a+h)+f^m(a)-Df^m(a)(h)).\end{aligned}$$

所以

$$\begin{aligned}&\lim_{h\to 0}\frac{|f(a+h)-f(a)-\lambda(h)|}{|h|}\\&\leqslant\lim_{h\to 0}\sum_{i=1}^{m}\frac{|f^i(a+h)-f^i(a)-Df^i(a)(h)|}{|h|}\\&=0.\end{aligned}$$

另一方面，如f在a处可微，则由(2)与定理2-2，$f^i=\pi^i\circ f$在a处可微.

(4) 由(2)推得.

(5) 令$\lambda(x,y)=bx+ay$. 那么

$$\lim_{(h,k)\to 0}\frac{|p(a+h,b+k)-p(a,b)-\lambda(h,k)|}{|(h,k)|}$$

$$=\lim_{(h,k)\to 0}\frac{|hk|}{|(h,k)|}.$$

现在

$$|hk| \leqslant \begin{cases} |h|^2, \text{如} |k| \leqslant |h|, \\ |k|^2, \text{如} |h| \leqslant |k|. \end{cases}$$

因此 $|hk| \leqslant |h|^2 + |k|^2$. 所以

$$\frac{|hk|}{|(h,k)|} \leqslant \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2},$$

因而

$$\lim_{(h,k)\to 0} \frac{|hk|}{|(h,k)|} = 0. \blacksquare$$

推论 2-4　如 $f,g:\mathbf{R}^n \to \mathbf{R}$ 在 a 处可微，则

$$\begin{aligned} D(f+g)(a) &= Df(a) + Dg(a), \\ D(f\cdot g)(a) &= g(a)Df(a) + f(a)Dg(a). \end{aligned}$$

此外，如果 $g(a) \neq 0$，则

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}.$$

证　我们将证明第一式而把其余的留给读者．因为 $f+g = s\circ(f,g)$，我们有

$$\begin{aligned} D(f+g)(a) &= Ds(f(a),g(a))\circ D(f,g)(a) \\ &= s\circ(Df(a),Dg(a)) \\ &= Df(a) + Dg(a). \blacksquare \end{aligned}$$

下面这样的函数 $f:\mathbf{R}^n \to \mathbf{R}^m$ 的可微性现在得到了保证，其各分量函数可以从函数 π^i（它们是线性变换）以及我们在初等微积分中早已会求导的函数经过加法、乘法、除法和复合而获得．但是，求 $Df(x)$ 或 $f'(x)$ 可能是一项相当艰巨的工作．例如，设 $f:\mathbf{R}^2 \to \mathbf{R}$ 定义为 $f(x,y) = \sin(xy^2)$. 因为 $f = \sin\circ(\pi^1\cdot[\pi^2]^2)$，我们有

$$\begin{aligned} f'(a,b) &= \sin'(ab^2)\cdot[b^2(\pi^1)'(a,b)+a([\pi^2]^2)'(a,b)] \\ &= \sin'(ab^2)\cdot[b^2(\pi^1)'(a,b)+2ab(\pi^2)'(a,b)] \\ &= (\cos(ab^2))\cdot[b^2(1,0)+2ab(0,1)] \\ &= (b^2\cos(ab^2),2ab\cos(ab^2)). \end{aligned}$$

幸而我们很快将会发现计算f'的一种简单得多的方法.

习题

2-10. 利用本节定理求以下的f':

(a) $f(x,y,z)=x^y$.

(b) $f(x,y,z)=(x^y,z)$.

(c) $f(x,y)=\sin(x\sin y)$.

(d) $f(x,y,z)=\sin(x\sin(y\sin z))$.

(e) $f(x,y,z)=x^{y^z}$.

(f) $f(x,y,z)=x^{y+z}$.

(g) $f(x,y,z)=(x+y)^z$.

(h) $f(x,y)=\sin(xy)$.

(i) $f(x,y)=[\sin(xy)]^{\cos 3}$

(j) $f(x,y)=(\sin(xy),\sin(x\sin y),x^y)$.

2-11. 求以下的f'（其中$g:\mathbf{R}\to\mathbf{R}$是连续的）:

(a) $f(x,y)=\int_a^{x+y}g$.

(b) $f(x,y)=\int_a^{x\cdot y}g$.

(c) $f(x,y,z)=\int_{xy}^{\sin(x\sin(y\sin z))}g$.

2-12. 一函数$f:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}^p$，如果对$x,x_1,x_2\in\mathbf{R}^n$，$y,y_1,y_2\in\mathbf{R}^m$以及$a\in\mathbf{R}$，我们有

$$\begin{aligned} f(ax,y)&=af(x,y)=f(x,ay), \\ f(x_1+x_2,y)&=f(x_1,y)+f(x_2,y), \\ f(x,y_1+y_2)&=f(x,y_1)+f(x,y_2), \end{aligned}$$

则称f是**双线性的**.

(a) 求证若f是双线性的，则

$$\lim_{(h,k)\to 0}\frac{|f(h,k)|}{|(h,k)|}=0.$$

(b) 求证 $Df(a,b)(x,y)=f(a,y)+f(x,b)$.

(c) 证明定理 2-3 中 $Dp(a,b)$ 的公式是(b)的一特殊情况.

2-13. 定义 $IP:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 为 $IP(x,y)=\langle x,y\rangle$.

(a) 求 $D(IP)(a,b)$ 与 $(IP)'(a,b)$.

(b) 如f, $g:\mathbf{R}\to\mathbf{R}^n$ 可微且 $h:\mathbf{R}\to\mathbf{R}$ 定义为 $h(t)=\langle f(t),g(t)\rangle$，证明

$$h'(a)=\langle f'(a)^{\mathrm{T}},g(a)\rangle+\langle f(a),g'(a)^{\mathrm{T}}\rangle.$$

(注意$f'(a)$ 是一个 $n\times 1$ 矩阵；其转置矩阵$f'(a)^{\mathrm{T}}$ 是一个 $1\times n$ 矩阵，我们把它看作 $\mathbf{R}^n$ 的元.)

(c) 如$f:\mathbf{R}\to\mathbf{R}^n$ 可微且对一切 t，$|f(t)|=1$，证明 $\langle f'(t)^{\mathrm{T}},f(t)\rangle=0$.

(d) 举出一可微函数$f:\mathbf{R}\to\mathbf{R}$ 使得由 $|f|(t)=|f(t)|$ 定义的函数 $|f|$ 不可微.

2-14. 设 $E_i(i=1,\cdots,k)$ 是各维数不必相同的欧氏空间. 一函数 $f:E_1\times\cdots\times E_k\to\mathbf{R}^p$ 称为是**重线性的**，如果对于每个选定的 $x_j\in E_j(j\neq i)$，由 $g(x)=f(x_1,\cdots,x_{i-1},x,x_{i+1},\cdots,x_k)$ 定义的函数 $g:E_i\to\mathbf{R}^p$ 是一个线性变换.

(a) 如果f是重线性的而且 $i\neq j$，证明对于 $h=(h_1,\cdots,h_k)$，其中 $h_l\in E_l$，我们有

$$\lim_{h\to 0}\frac{|f(a_1,\cdots,h_i,\cdots,h_j,\cdots,a_k)|}{|h|}=0.$$

提示：如果 $g(x,y)=f(a_1,\cdots,x,\cdots,y,\cdots,a_k)$，则 g 是双线性的.

(b) 求证

$$Df(a_1,\cdots,a_k)(x_1,\cdots,x_k)=\sum_{i=1}^{k}f(a_1,\cdots,a_{i-1},x_i,a_{i+1},\cdots,a_k).$$

2-15. 把一个 $n\times n$ 矩阵的每一列视作 $\mathbf{R}^n$ 的一元，从而把矩阵本身当作 n 重乘积 $\mathbf{R}^n\times\cdots\times\mathbf{R}^n$ 中的一个点.

(a) 求证 $\det:\mathbf{R}^n\times\cdots\times\mathbf{R}^n\to\mathbf{R}$ 可微且

$$D(\det)(a_1,\cdots,a_n)(x_1,\cdots,x_n)=\sum_{i=1}^{n}\det\begin{bmatrix}a_1\\ \vdots\\ x_i\\ \vdots\\ a_n\end{bmatrix}.$$

(b) 如果 $a_{ij}:\mathbf{R}\to\mathbf{R}$ 可微而 $f(t)=\det(a_{ij}(t))$，试证

$$f'(t)=\sum_{j=1}^{n}\det\begin{bmatrix} a_{11}(t),\cdots,a_{1n}(t)\\ \vdots \qquad \vdots \\ a'_{j1}(t),\cdots,a'_{jn}(t)\\ \vdots \qquad \vdots \\ a_{n1}(t),\cdots,a_{nn}(t)\end{bmatrix}.$$

(c) 如对一切 t，$\det(a_{ij}(t))\neq 0$，且 $b_1,\cdots,b_n:\mathbf{R}\to\mathbf{R}$ 都是可微的，又设 $s_1,\cdots,s_n:\mathbf{R}\to\mathbf{R}$ 是这样的一些函数，使得 $s_1(t),\cdots,s_n(t)$ 是方程组

$$\sum_{j=1}^{n}a_{ji}(t)s_j(t)=b_i(t),i=1,\cdots,n$$

的解．求证 s_i 可微并求出 $s'_i(t)$.

2-16. 设 $f:\mathbf{R}^n\to\mathbf{R}^n$ 可微且有可微的逆 $f^{-1}:\mathbf{R}^n\to\mathbf{R}^n$. 证明 $(f^{-1})'(a)=[f'(f^{-1}(a))]^{-1}$. 提示：$f\circ f^{-1}(x)=x$.

2.3 偏 导 数

我们从讨论“每次对一个变元”求导数的问题开始．如 $f:\mathbf{R}^n\to\mathbf{R}$ 且 $a\in\mathbf{R}^n$，如极限

$$\lim_{h\to 0}\frac{f(a^1,\cdots,a^i+h,\cdots,a^n)-f(a^1,\cdots,a^n)}{h}$$

存在，就记作 $D_if(a)$，称为 f 在 a 点的**偏导数**．注意 $D_if(a)$ 是某函数的通常导数，这很重要．实际上，如 $g(x)=f(a^1,\cdots,x,\cdots,a^n)$，则 $D_if(a)=g'(a^i)$．这表明，$D_if(a)$ 是 f 的图形和平面 $x^j=a^j(j\neq i)$ 的交线在 $(a,f(a))$ 点切线的斜率(图 2-1)．这也表明，计算 $D_if(a)$ 是我们早已会做的问题．如 $f(x^1,\cdots,x^n)$ 已由含有 $x^1,\cdots,x^n$ 的某公式给出，则我们可这样来求 $D_if(x^1,\cdots,x^n)$，即把所有 $x^j(j\neq i)$ 都看作常数，而对所得的 x^i 的函数对 x^i 求导．例如，如果 $f(x,y)=\sin(xy^2)$，则 $D_1f(x,y)=y^2\cos(xy^2)$，$D_2f(x,y)=2xy\cos(xy^2)$．又如，$f(x,y)=x^y$，则 $D_1f(x,y)=yx^{y-1}$，$D_2f(x,y)=x^y\ln x$.

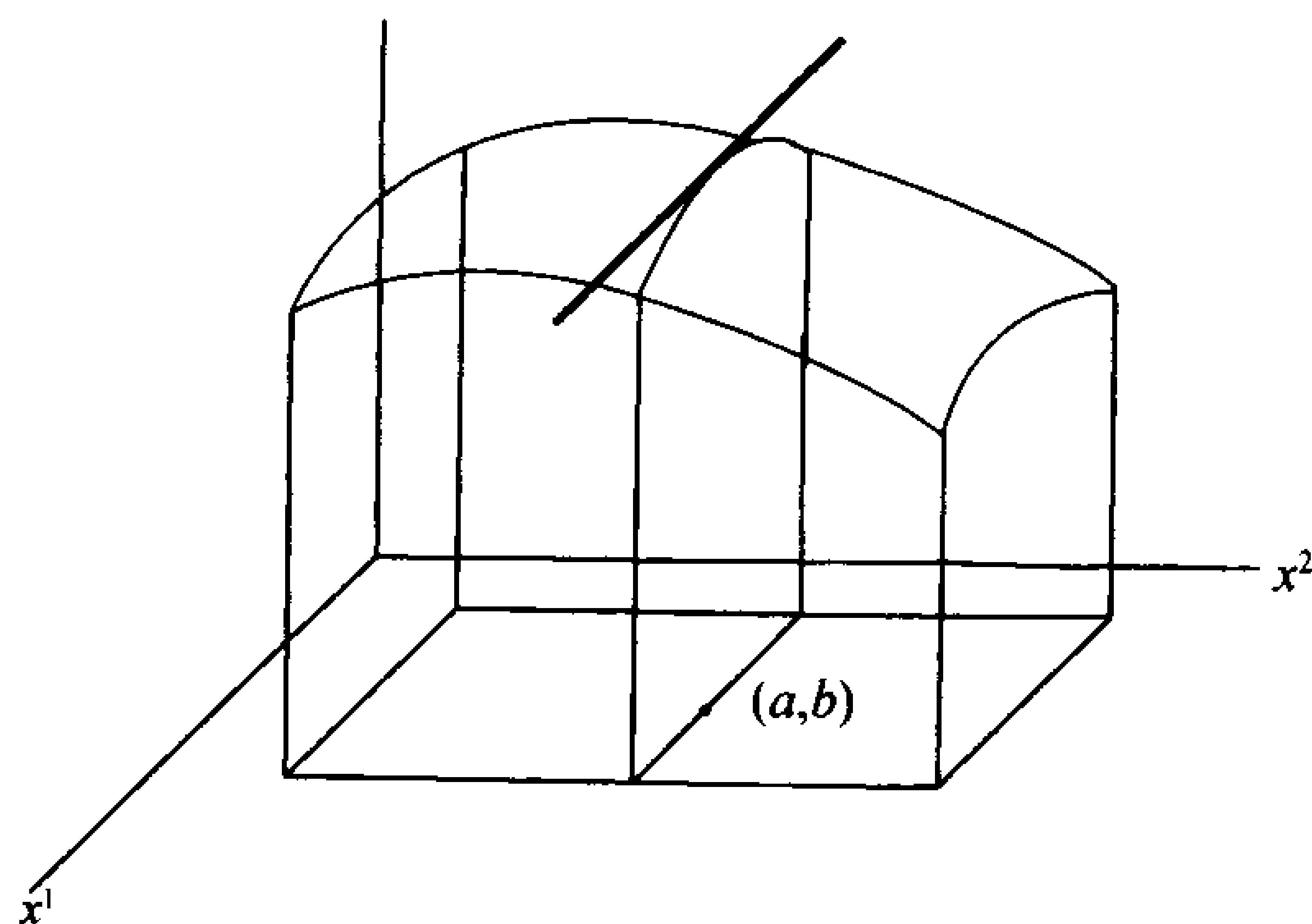


图 2-1

稍经练习(例如，做本节末的习题)，就和已经会计算的通常导数一样，也会很容易地计算 $D_i f$.

如果对一切 $x \in \mathbf{R}^n$, $D_i f(x)$ 存在，我们便得一个函数 $D_i f: \mathbf{R}^n \to \mathbf{R}$. 这个函数在 x 点的第 j 个偏导数，也就是 $D_j(D_i f)(x)$，常常记作 $D_{i,j}f(x)$. 注意这个记号把 i 与 j 的次序颠倒了. 实际上，这个次序通常是没有关系的，因为绝大多数函数（在习题中给出个例外）满足 $D_{i,j}f = D_{j,i}f$. 有好些细致的定理保证这个等式. 下面这个定理已经完全够用了. 我们把它的陈述放在这里而把证明放在后面（习题 3-28).

定理 2-5 如 $D_{i,j}f$ 与 $D_{j,i}f$ 在包含 a 的一开集中连续，则

$$D_{i,j}f(a) = D_{j,i}f(a).$$

函数 $D_{i,j}f$ 叫作 f 的**二阶（混合）偏导数**. 高阶（混合）偏导数可用明显的方式来定义. 显然定理 2-5 能用来证明在适当条件下高阶混合偏导数的相应等式. 如 f 有一切阶的偏导数，则 $D_{i_1,\cdots,i_k}f$ 中 $i_1,\cdots,i_k$ 的次序是完全无所谓的. 具有这种性质的函数称为 C^∞ 函数. 在以下各章中，为方便起见，经常仅限于讨论 C^∞ 函数.

在下节，将用偏导数来求导数. 它们还有另外一个重要的用处——求函数的极大值和极小值.

定理 2-6 设 $A \subset \mathbf{R}^n$. 如 $f:A \to \mathbf{R}$ 在 A 的内域中点 a 处达到极大（或极小），且 $D_i f(a)$ 存在，则 $D_i f(a) = 0$.

证 设 $g_i(x) = f(a^1, \cdots, x, \cdots, a^n)$. 显然 g_i 在 a^i 处有极大值（或极小值），且 g_i 在包含 a^i 的一开区间中有定义．因此 $0 = g_i'(a^i) = D_i f(a)$. ▌

提醒读者，定理 2-6 的逆即使当 $n = 1$ 时也不成立（如 $f:\mathbf{R} \to \mathbf{R}$ 由 $f(x) = x^3$ 定义，则 $f'(0) = 0$，但 0 点甚至不是一个局部极大值或极小值）. 如 $n > 1$，定理2-6的逆可以在一种更为奇特的方式下不再为真．例如，设 $f:\mathbf{R}^2 \to \mathbf{R}$ 由 $f(x, y) = x^2 - y^2$ 定义(图 2-2). 则因 g_1 在 0 处有一极小值，故 $D_1 f(0,0) = 0$；而因 g_2 在 0 处有一个极大值，故 $D_2 f(0,0) = 0$. 显然 (0,0) 既不是相对极大点也不是相对极小点．

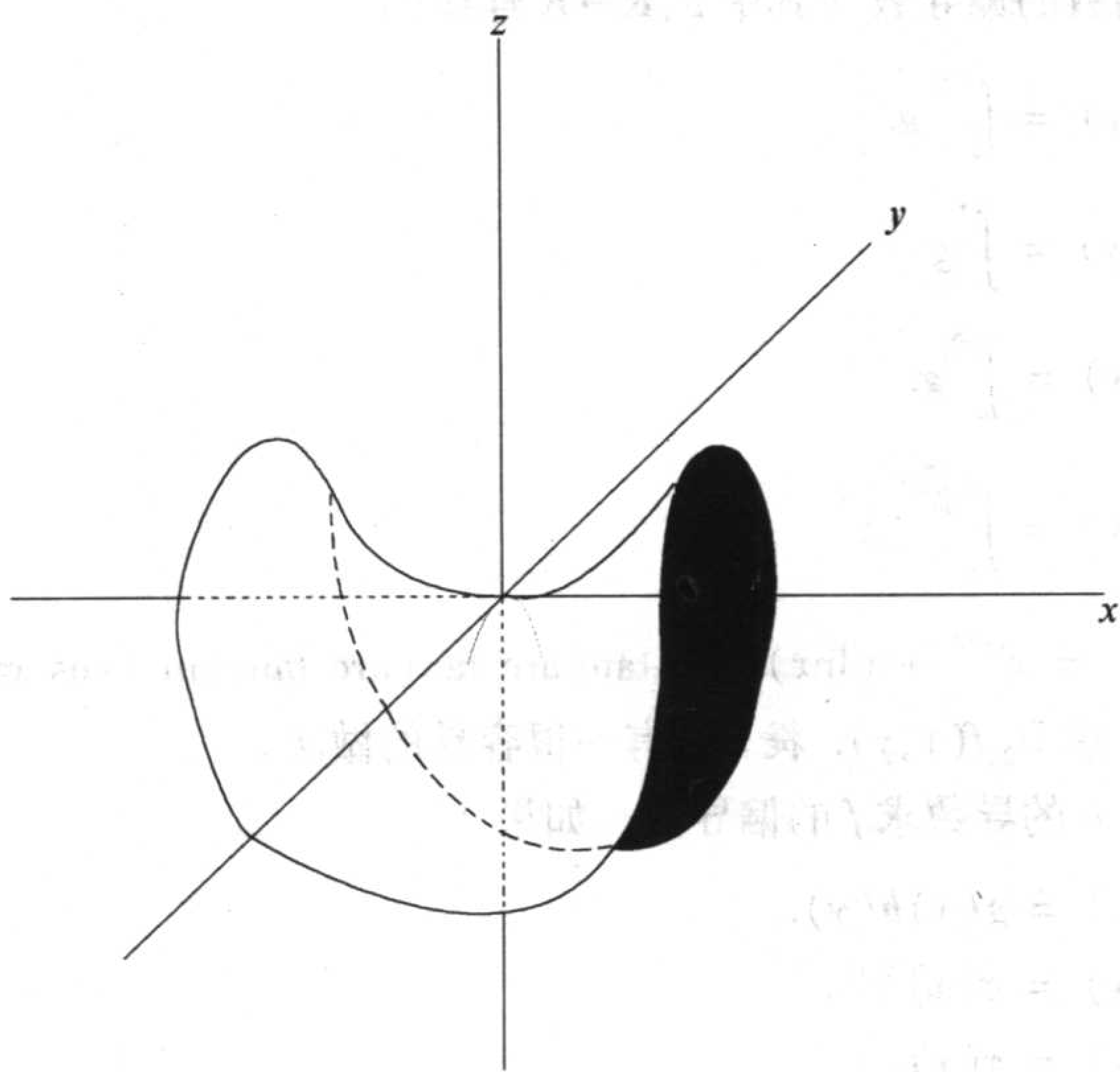


图 2-2

如用定理 2-6 来寻求 f 在 A 上的最大值或最小值，那么还必须单另检查 f 在边界点上的值——这是一件可怕的事情，因为 A 的边界可能是整个 A. 习题2-27指明一种做法，习题 5-16 陈述了一个经常可用

的好方法.

习题

2-17. 求下列函数的偏导数:

(a) $f(x,y,z) = x^y$.

(b) $f(x,y,z) = z$.

(c) $f(x,y) = \sin(x \sin y)$.

(d) $f(x,y,z) = \sin(x \sin(y \sin z))$.

(e) $f(x,y,z) = x^{y^z}$.

(f) $f(x,y,z) = x^{y+z}$.

(g) $f(x,y,z) = (x+y)^z$.

(h) $f(x,y) = \sin(xy)$.

(i) $f(x,y) = [\sin(xy)]^{\cos 3}$.

2-18. 求下列函数的偏导数（其中 $g:\mathbf{R}\to\mathbf{R}$ 连续）:

(a) $f(x,y) = \int_a^{x+y} g$.

(b) $f(x,y) = \int_y^x g$.

(c) $f(x,y) = \int_a^{xy} g$.

(d) $f(x,y) = \int_a^{(\int_b^y g)} g$.

2-19. 如 $f(x,y) = x^{x^{x^{x^y}}} + (\ln x)(\arctan(\arctan(\arctan(\sin(\cos xy) - \ln(x+y)))))$, 求 $D_2 f(1,y)$. 提示: 有一很容易的做法.

2-20. 通过 g 与 h 的导数求 f 的偏导数, 如果

(a) $f(x,y) = g(x)h(y)$.

(b) $f(x,y) = g(x)^{h(y)}$.

(c) $f(x,y) = g(x)$.

(d) $f(x,y) = g(y)$.

(e) $f(x,y) = g(x+y)$.

2-21.* 设 $g_1, g_2:\mathbf{R}^2\to\mathbf{R}$ 连续. 定义 $f:\mathbf{R}^2\to\mathbf{R}$ 为

$$f(x,y) = \int_0^x g_1(t,0)\,dt + \int_0^y g_2(x,t)\,dt.$$

(a) 证明 $D_2 f(x,y) = g_2(x,y)$.

(b) f应怎样定义使得 $D_1 f(x,y) = g_1(x,y)$?

(c) 求一函数f:$\mathbf{R}^2 \to \mathbf{R}$ 使得 $D_1 f(x,y) = x$, $D_2 f(x,y) = y$. 再求一个使得 $D_1 f(x,y) = y$, $D_2 f(x,y) = x$.

2-22.* 如f:$\mathbf{R}^2 \to \mathbf{R}$ 且 $D_2 f = 0$, 证明f与第二个变元无关. 如果 $D_1 f = D_2 f = 0$, 证明f是常数.

2-23.* 设 $A = \{(x,y) \in \mathbf{R}^2 : x < 0$ 或者 $x \geqslant 0$ 且 $y \neq 0\}$.

(a) 如果f: $A \to \mathbf{R}$ 且 $D_1 f = D_2 f = 0$, 证明f是一个常数. 提示: 注意, A 中任两点可用一串直线段联结, 每一段平行于坐标轴之一.

(b) 求一函数f:$A \to \mathbf{R}$ 使得 $D_2 f = 0$, 但f不是与第二变元无关.

2-24. 定义f:$\mathbf{R}^2 \to \mathbf{R}$ 为

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

(a) 试证: 对一切 x, $D_2 f(x,0) = x$; 对一切 y, $D_1 f(0,y) = -y$.

(b) 试证 $D_{1,2} f(0,0) \neq D_{2,1} f(0,0)$.

2-25.* 定义f:$\mathbf{R} \to \mathbf{R}$ 为

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

证明f是一 C^∞ 函数, 且对一切 $i, f^{(i)}(0) = 0$. 提示: 极限

$$f'(0) = \lim_{h \to 0} \frac{e^{-h^{-2}}}{h} = \lim_{h \to 0} \frac{1/h}{e^{h^{-2}}}$$

可用罗必达法则计算. 对 $x \neq 0$ 求 $f'(x)$ 非常容易, 然后 $f'(0) = \lim\limits_{h \to 0} f'(h)/h$ 可用罗必达法则求得.

2-26.* 设

$$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1), \\ 0 & x \notin (-1,1). \end{cases}$$

(a) 证明f:$\mathbf{R} \to \mathbf{R}$ 是一 C^∞ 函数, 它在 $(-1, 1)$ 内为正, 在其他处为 0.

(b) 证明存在一 C^∞ 函数 g:$\mathbf{R} \to [0,1]$ 使得当 $x \leqslant 0$ 时 $g(x) = 0$, 当 $x \geqslant \varepsilon$ 时 $g(x) = 1$. 提示: 如果f是一个 C^∞ 函数, 在 $(0,\varepsilon)$ 内为正, 在其他处为 0, 令

$$g(x) = \int_0^x f \Big/ \int_0^\varepsilon f.$$

(c) 如 $a \in \mathbf{R}^n$, 定义 $g:\mathbf{R}^n \to \mathbf{R}$ 为

$$g(x) = f([x^1 - a^1]/\varepsilon) \cdot \cdots \cdot f([x^n - a^n]/\varepsilon).$$

证明 g 是一个 C^∞ 函数，它在

$$(a^1 - \varepsilon, a^1 + \varepsilon) \times \cdots \times (a^n - \varepsilon, a^n + \varepsilon)$$

内为正，在其他处为零.

(d) 如 $A \subset \mathbf{R}^n$ 是开的且 $C \subset A$ 是紧的，证明存在一个非负 C^∞ 函数 $f:A \to \mathbf{R}$ 使当 $x \in C$ 时 $f(x) > 0$, 而在含于 A 内的某闭集之外 $f=0$.

(e) 证明可以选取这样的 f 使得 $f:A \to [0,1]$, 且对 $x \in C$, $f(x) = 1$. 提示:如果(d)中的函数 f 当 $x \in C$ 时 $f(x) \geq \varepsilon$, 考察 $g \circ f$, 其中 g 是 (b) 中的函数.

2-27. 定义 $g, h: \{x \in \mathbf{R}^2 : |x| \leq 1\} \to \mathbf{R}^3$ 为

$$g(x,y) = (x, y, \sqrt{1 - x^2 - y^2}),$$
$$h(x,y) = (x, y, -\sqrt{1 - x^2 - y^2}).$$

证明 f 在 $\{x \in \mathbf{R}^3 : |x| = 1\}$ 上的最大值或者是 $\{x \in \mathbf{R}^2 : |x| \leq 1\}$ 上 $f \circ g$ 的最大值，或者是它上面 $f \circ h$ 的最大值.

2.4 导 数

比较过习题 2-10 和 2-17 的读者可能已经猜到下面的结论:

定理 2-7 如果 $f:\mathbf{R}^n \to \mathbf{R}^m$ 在 a 点可微，则对于 $1 \leq i \leq m, 1 \leq j \leq n$, $D_j f^i(a)$ 存在，且 $f'(a)$ 是 $m \times n$ 矩阵 $(D_j f^i(a))$.

证 先假定 $m=1$, 故 $f:\mathbf{R}^n \to \mathbf{R}$. 用 $h(x) = (a^1, \cdots, x, \cdots, a^n)$ 定义 $h:\mathbf{R} \to \mathbf{R}^n$, 其中 x 在第 j 个位置. 则 $D_j f(a) = (f \circ h)'(a^j)$. 因此，由定理 2-2,

$$(f \circ h)'(a^j) = f'(a) \cdot h'(a^j)$$

$$= f'(a) \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{第} j \text{个位置}.$$

因为 $(f \circ h)'(a^j)$ 有惟一的元素 $D_j f(a)$，这就表明 $D_j f(a)$ 存在且就是 $1 \times n$矩阵$f'(a)$ 的第j个元素.

因为由定理2-3 每一f^i可微，且$f'(a)$ 的第i行是 $(f^i)'(a)$，所以现在本定理就对任意的 m 都成立. ▌

在习题中有几个例子表明定理2-7 的逆不成立. 但若添加一个假设，它还是对的.

定理 2-8 *如f: $\mathbf{R}^n \to \mathbf{R}^m$，则若所有 $D_j f^i(x)$ 在包含 a 的一开集中存在且每一函数 $D_j f^i$在 a 点连续，则 $Df(a)$存在.（这样的函数f称为在 a 点连续可微.）*

证 和定理2-7 的证明一样，只要考察 $m=1$ 的情况就够了，所以f:$\mathbf{R}^n \to \mathbf{R}$. 于是

$$\begin{aligned} f(a+h) - f(a) &= f(a^1+h^1, a^2, \cdots, a^n) - f(a^1, \cdots, a^n) \\ &+ f(a^1+h^1, a^2+h^2, a^3, \cdots, a^n) - f(a^1+h^1, a^2, \cdots, a^n) + \cdots \\ &+ f(a^1+h^1, \cdots, a^n+h^n) - f(a^1+h^1, \cdots, a^{n-1}+h^{n-1}, a^n). \end{aligned}$$

回想 $D_1 f$是由 $g(x) = f(x, a^2, \cdots, a^n)$ 定义的函数g 的导数. 对g 应用中值定理，便得

$$f(a^1+h^1, a^2, \cdots, a^n) - f(a^1, \cdots, a^n) = h^1 \cdot D_1 f(b_1, a^2, \cdots, a^n),$$

这里 b_1 是 a^1 与 a^1+h^1 间的某数. 同样，在和式中第 i 项等于（对某 c_i）

$$h^i \cdot D_i f(a^1+h^1, \cdots, a^{i-1}+h^{i-1}, b_i, \cdots, a^n) = h^i D_i f(c_i).$$

于是，

$$\lim_{h \to 0} \frac{\left| f(a+h) - f(a) - \sum_{i=1}^{n} D_i f(a) \cdot h^i \right|}{|h|}$$

$$
\begin{aligned}
&= \lim_{h\to 0}\frac{\left|\sum_{i=1}^{n}[D_i f(c_i) - D_i f(a)]\cdot h^i\right|}{|h|}\\
&\leqslant \lim_{h\to 0}\sum_{i=1}^{n}|D_i f(c_i) - D_i f(a)|\cdot\frac{|h^i|}{|h|}\\
&\leqslant \lim_{h\to 0}\sum_{i=1}^{n}|D_i f(c_i) - D_i f(a)|\\
&= 0,
\end{aligned}
$$

因为 $D_i f$ 在 a 点连续. ▌

虽然在证明定理2-7时应用了锁链规则，但能容易地把它去掉. 对可微函数有定理2-8，对它们的导数有定理2-7，所以锁链规则看来似乎是多余的. 但是，它有一个关于偏导数的极端重要的推论.

定理2-9 设 $g_1,\cdots,g_m:\mathbf{R}^n\to\mathbf{R}$ 在 a 点连续可微，并设 $f:\mathbf{R}^m\to\mathbf{R}$ 在 $(g_1(a),\cdots,g_m(a))$ 点连续可微. 用 $F(x)=f(g_1(x),\cdots,g_m(x))$ 定义 $F:\mathbf{R}^n\to\mathbf{R}$. 则

$$D_iF(a) = \sum_{j=1}^{m} D_jf(g_1(a),\cdots,g_m(a))\cdot D_ig_j(a).$$

证 函数 F 正好是复合函数 $f\circ g$，其中 $g=(g_1,\cdots,g_m)$. 因 g_i 在 a 点连续可微，故由定理2-8得知 g 在 a 点可微. 故由定理2-2，得

$$
\begin{aligned}
F'(a) &= f'(g(a))\cdot g'(a)\\
&= (D_1f(g(a)),\cdots,D_mf(g(a)))\cdot\\
&\begin{pmatrix} D_1g_1(a),\cdots,D_ng_1(a)\\ \vdots \qquad\qquad \vdots\\ D_1g_m(a),\cdots,D_ng_m(a)\end{pmatrix}.
\end{aligned}
$$

但 $D_iF(a)$ 是此式左方第 i 个元素，而 $\sum_{j=1}^{m} D_jf(g_1(a),\cdots,g_m(a))\cdot D_ig_j(a)$ 是右方第 i 个元素. ▌

定理2-9也常常称为锁链规则，但较定理2-2为弱，因为 g 或 f 可微并不需要 g_i 或 f 连续可微（见习题2-32）. 绝大多数要求利用定

理 2-9 的规则的计算都是十分直接的．对于由

$$F(x,y)=f(g(x,y),h(x),k(y))$$

定义的 $F:\mathbf{R}^2\to\mathbf{R}$，其中 $h,k:\mathbf{R}\to\mathbf{R}$，要求稍为细致些．为要应用定理 2-9，用

$$\bar{h}(x,y)=h(x),\ \bar{k}(x,y)=k(y)$$

定义 $\bar{h},\bar{k}:\mathbf{R}^2\to\mathbf{R}$. 于是

$$\begin{aligned} D_1\bar{h}(x,y)&=h'(x) & D_2\bar{h}(x,y)&=0,\\ D_1\bar{k}(x,y)&=0 & D_2\bar{k}(x,y)&=k'(y), \end{aligned}$$

我们可以写

$$F(x,y)=f(g(x,y),\bar{h}(x,y),\bar{k}(x,y)).$$

令 $a=(g(x,y),h(x),k(y))$，便得

$$\begin{aligned} D_1F(x,y)&=D_1f(a)\cdot D_1g(x,y)+D_2f(a)\cdot h'(x),\\ D_2F(x,y)&=D_1f(a)\cdot D_2g(x,y)+D_3f(a)\cdot k'(y). \end{aligned}$$

当然，没有必要真地写出函数 $\bar{h}$ 与 $\bar{k}$.

习题

2-28. 求下列函数的偏导数的表达式：

(a) $F(x,y)=f(g(x)k(y),g(x)+h(y))$.

(b) $F(x,y,z)=f(g(x+y),h(y+z))$.

(c) $F(x,y,z)=f(x^y,y^z,z^x)$.

(d) $F(x,y)=f(x,g(x),h(x,y))$.

2-29. 设 $f:\mathbf{R}^n\to\mathbf{R}$. 对 $x\in\mathbf{R}^n$，极限

$$\lim_{t\to 0}\frac{f(a+tx)-f(a)}{t},$$

如果存在的话，记作 $D_xf(a)$，并称之为 f 在 a 处沿方向 x 的**方向导数**．

(a) 试证 $D_{e_i}f(a)=D_if(a)$.

(b) 试证 $D_{tx}f(a)=tD_xf(a)$.

(c) 如 f 在 a 处可微，试证 $D_xf(a)=Df(a)(x)$，从而 $D_{x+y}f(a)=D_xf(a)$

$+ D_y f(a)$.

2-30. 设f如习题2-4中所定义．证明对一切x, $D_x f(0,0)$存在；但若$g \neq 0$, 则对一切x与y, $D_{x+y} f(0,0) = D_x f(0,0) + D_y f(0,0)$不真．

2-31. 设$f: \mathbf{R}^2 \to \mathbf{R}$如习题1-26中所定义．试证：虽然$f$在$(0,0)$处甚至是不连续的，但$D_x f(0,0)$对一切$x$存在．

2-32. (a) 设$f: \mathbf{R} \to \mathbf{R}$定义为

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

证明f在0点可微，但f'在0点不连续．

(b) 设$f: \mathbf{R}^2 \to \mathbf{R}$定义为

$$f(x,y) = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{\sqrt{x^2 + y^2}} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

证明f在$(0,0)$点可微，但$D_i f$在$(0,0)$点不连续．

2-33. 求证：$D_1 f^i$在a点的连续性可以从定理2-8的假设中去掉．

2-34. 一个函数$f: \mathbf{R}^n \to \mathbf{R}$, 如对一切$x$，都有$f(tx) = t^m f(x)$，就称为$m$次齐次的．如$f$还是可微的，试证

$$\sum_{i=1}^{n} x^i D_i f(x) = m f(x).$$

提示：如果$g(t) = f(tx)$，求$g'(1)$.

2-35. 如$f: \mathbf{R}^n \to \mathbf{R}$可微且$f(0) = 0$, 求证存在$g_i: \mathbf{R}^n \to \mathbf{R}$使得

$$f(x) = \sum_{i=1}^{n} x^i g_i(x).$$

提示：如果$h_x(t) = f(tx)$，则$f(x) = \int_0^1 h'_x(t)\,dt$.

2.5 反 函 数

设$f: \mathbf{R} \to \mathbf{R}$在包含$a$的一开集中连续可微且$f'(a) \neq 0$. 若$f'(a) > 0$, 则存在一个包含$a$的开区间$V$使对$x \in V$, 有$f'(x) > 0$；

又若$f'(a)<0$，则有类似的命题．于是f在V中递增（或递减），所以是1-1的，而且存在定义在包含$f(a)$的某开区间W上的反函数f^{-1}．此外不难证明f^{-1}是可微的，而且对$y\in W$，有

$$(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}.$$

对高维类似的讨论就复杂多了．但其结果（定理2-11）却非常重要．先讲一个简单引理．

引理2-10 设$A\subset\mathbf{R}^n$是一个矩形，并设$f:A\to\mathbf{R}^n$连续可微．如存在一数M对A的内域中的一切x有$|D_jf^i(x)|\leqslant M$，则对所有的$x,y\in A$，有

$$|f(x)-f(y)|\leqslant n^2M|x-y|.$$

证 我们有

$$\begin{aligned}f^i(y)-f^i(x)=\sum_{j=1}^{n}[&f^i(y^1,\cdots,y^j,x^{j+1},\cdots,x^n)\\&-f^i(y^1,\cdots,y^{j-1},x^j,\cdots,x^n)].\end{aligned}$$

应用中值定理，便得（对某z_{ij}）

$$\begin{aligned}f^i(y^1,\cdots,y^j,x^{j+1},\cdots,x^n)&-f^i(y^1,\cdots,y^{j-1},x^j,\cdots,x^n)\\&=(y^j-x^j)\cdot D_jf^i(z_{ij}),\end{aligned}$$

等式右边的绝对值小于或等于$M\cdot|y^j-x^j|$．于是，

$$|f^i(y)-f^i(x)|\leqslant\sum_{j=1}^{n}|y^j-x^j|\cdot M\leqslant nM|y-x|$$

因为每一$|y^j-x^j|\leqslant|y-x|$．最后，得

$$|f(y)-f(x)|\leqslant\sum_{i=1}^{n}|f^i(y)-f^i(x)|\leqslant n^2M\cdot|y-x|.\ \blacksquare$$

定理2-11（反函数定理） 设$f:\mathbf{R}^n\to\mathbf{R}^n$在包含$a$的一个开集中连续可微，且$\det f'(a)\neq0$．则存在包含$a$的一个开集$V$和包含$f(a)$的一个开集$W$，使得$f:V\to W$有一个连续可微的反函数$f^{-1}:W\to V$，而且对

一切 $y \in W$ 满足

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}.$$

证 设λ 是线性变换 $Df(a)$. 则因 $\det f'(a) \neq 0$, 故λ 是非退化的. 现在 $D(\lambda^{-1} \circ f)(a) = D(\lambda^{-1})(f(a)) \circ Df(a) = \lambda^{-1} \circ Df(a)$ 是恒等线性变换. 如定理对 $\lambda^{-1} \circ f$ 为真, 显然对f 也真. 所以我们一开始就可认为λ 是恒等变换. 于是一旦$f(a+h) = f(a)$, 就有

$$\frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = \frac{|h|}{|h|} = 1.$$

但

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

这表明, 对于任意地靠近 a 但不等于 a 的 x, 不能有$f(x) = f(a)$. 所以存在包含 a 在其内域的闭矩形 U, 使得:

1. $f(x) \neq f(a)$, 如 $x \in U$ 且 $x \neq a$.

因为f 在包含 a 的一个开集中连续可微, 还可认为

2. 对 $x \in U$, $\det f'(x) \neq 0$.

3. 对一切 i,j 以及 $x \in U$, $|D_j f^i(x) - D_j f^i(a)| < 1/2n^2$.

注意, 把(3)和引理 2-10 应用于 $g(x) = f(x) - x$, 就隐含着对 $x_1, x_2 \in U$ 有

$$|f(x_1) - x_1 - (f(x_2) - x_2)| \leqslant \frac{1}{2}|x_1 - x_2|.$$

因为

$$\begin{aligned} |x_1 - x_2| - |f(x_1) - f(x_2)| &\leqslant |f(x_1) - x_1 - (f(x_2) - x_2)| \\ &\leqslant \frac{1}{2}|x_1 - x_2|, \end{aligned}$$

便得到

4. 对一切 $x_1, x_2 \in U$, $|x_1 - x_2| \leqslant 2|f(x_1) - f(x_2)|$.

现在f（U的边界）是一个紧集，由(1)它不含$f(a)$（图2-3）．所以存在一个数$d>0$使当$x\in$（U的边界）时$|f(a)-f(x)|\geqslant d$. 令$W=\{y: |y-f(a)|<d/2\}$. 如$y\in W$且$x\in$（U的边界），则

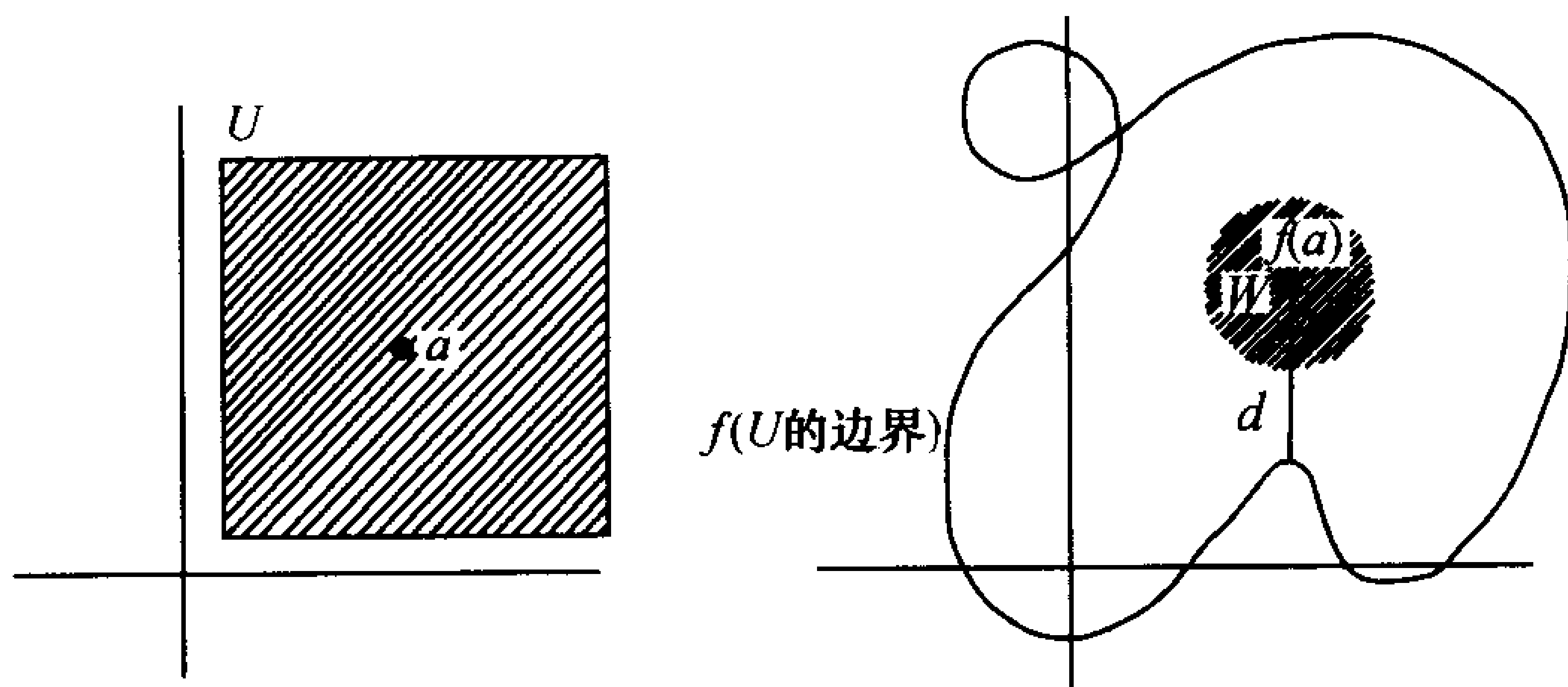


图2-3

5. $|y-f(a)|<|y-f(x)|$.

我们将证明，对任一个$y\in W$, 在U的内域中存在惟一的x使得$f(x)=y$. 为证此，考虑用

$$g(x)=|y-f(x)|^2=\sum_{i=1}^{n}(y^i-f^i(x))^2$$

定义的函数$g:U\to\mathbf{R}$. 这函数是连续的，所以在U上有一个最小值．如$x\in$（U的边界），则由(5)，有$g(a)<g(x)$. 所以g的最小值不会出现在U的边界上．由定理2-6，存在一点$x\in$（U的内域），使对一切j，$D_jg(x)=0$，也就是，

$$\sum_{i=1}^{n}2(y^i-f^i(x))\cdot D_jf^i(x)=0, \text{对一切}j.$$

由(2)，矩阵$(D_jf^i(x))$有非零的行列式．所以对一切i必有$y^i-f^i(x)=0$, 也就是$y=f(x)$. 这就证明了x的存在性．惟一性从(4)立得．

如$V=$（U的内域）$\cap f^{-1}(W)$，我们已经证明了函数$f:V\to W$有逆$f^{-1}:W\to V$. 可把（4）改写为

6. 对$y_1,y_2\in W$，有$|f^{-1}(y_1)-f^{-1}(y_2)|\leqslant 2|y_1-y_2|$．这就证明了$f^{-1}$连续．

剩下只有f^{-1}可微还未证．设$\mu = Df(x)$，我们将证明f^{-1}在$y = f(x)$处可微，有导数μ^{-1}．和定理2-2的证明中一样，对$x_1 \in V$，我们有

$$f(x_1) = f(x) + \mu(x_1 - x) + \varphi(x_1 - x),$$

其中

$$\lim_{x_1 \to x} \frac{|\varphi(x_1 - x)|}{|x_1 - x|} = 0.$$

所以

$$\mu^{-1}(f(x_1) - f(x)) = x_1 - x + \mu^{-1}(\varphi(x_1 - x)).$$

因为任意一个$y_1 \in W$都具有$f(x_1)$的形式（对某一$x_1 \in V$），这可以写成

$$f^{-1}(y_1) = f^{-1}(y) + \mu^{-1}(y_1 - y) - \mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y))),$$

所以只要证明

$$\lim_{y_1 \to y} \frac{|\mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y)))|}{|y_1 - y|} = 0.$$

因而（习题1-10）只要证明

$$\lim_{y_1 \to y} \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|} = 0.$$

现在

$$\begin{aligned} &\frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|} \\ &= \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|} \cdot \frac{|f^{-1}(y_1) - f^{-1}(y)|}{|y_1 - y|}. \end{aligned}$$

因为f^{-1}是连续的，当$y_1 \to y$时$f^{-1}(y_1) \to f^{-1}(y)$．所以第一因子趋于0．因为由(6)，第二因子小于2，故乘积也趋于0．▌

应当注意，即使$\det f'(a) = 0$，反函数f^{-1}也还是可能存在的．例如，若$f: \mathbf{R} \to \mathbf{R}$由$f(x) = x^3$定义，于是$f'(0) = 0$，但$f$有反函数

$f^{-1}(x)=\sqrt[3]{x}$. 然而有一件事可以肯定：如 $\det f'(a)=0$，则 f^{-1} 在 $f(a)$ 处必不可微．为证此，注意 $f\circ f^{-1}(x)=x$. 假若 f^{-1} 在 $f(a)$ 处可微，则锁链规则将给出 $f'(a)\cdot(f^{-1})'(f(a))=I$，因此 $\det f'(a)\cdot\det(f^{-1})'(f(a))=1$，此与 $\det f'(a)=0$ 相矛盾．

习题

2-36. * 设 $A\subset\mathbf{R}^n$ 是一开集，$f:A\to\mathbf{R}^n$ 是连续可微的 1-1 函数，使对一切 x 皆有 $\det f'(x)\neq 0$. 证明 $f(A)$ 是开集而且 $f^{-1}:f(A)\to A$ 可微．再证明，对任一开集 $B\subset A$，$f(B)$ 是开的．

2-37. (a) 设 $f:\mathbf{R}^2\to\mathbf{R}$ 是一连续可微函数，而且 $D_1f(x,y)$，$D_2f(x,y)$[1] 不同时为 0. 试证 f 不是 1-1 的．提示：例如，如果对某开集 A 中的一切 (x,y)，$D_1f(x,y)\neq 0$，考察由 $g(x,y)=(f(x,y),y)$ 定义的 $g:A\to\mathbf{R}^2$.

(b) 将这结果推广到连续可微函数 $f:\mathbf{R}^n\to\mathbf{R}^m$ 而 $m<n$ 的情况．

2-38. (a) 如 $f:\mathbf{R}\to\mathbf{R}$ 满足 $f'(a)\neq 0$（对一切 $a\in\mathbf{R}$），证明 f 在整个 $\mathbf{R}$ 上是1-1 的．

(b) 用 $f(x,y)=(e^x\cos y,e^x\sin y)$ 定义 $f:\mathbf{R}^2\to\mathbf{R}^2$. 证明．对一切 (x,y)，$\det f'(x,y)\neq 0$ 但 f 不是 1-1 的．

2-39. 利用由

$$f(x)=\begin{cases}\dfrac{x}{2}+x^2\sin\dfrac{1}{x} & x\neq 0,\\ 0 & x=0,\end{cases}$$

定义的函数 $f:\mathbf{R}\to\mathbf{R}$，证明导数的连续性条件不能从定理 2-11 的假设中去掉.

2.6 隐 函 数

考察由 $f(x,y)=x^2+y^2-1$ 定义的函数 $f:\mathbf{R}^2\to\mathbf{R}$. 如果我们选择 (a,b) 使 $f(a,b)=0$ 且 $a\neq 1,-1$，则存在（图 2-4）包含 a 的开区间

1. 原书并没有加这个条件，因此是错的，详见习题解答．——译者注

A 和包含 b 的开区间 B，具有下列性质：如果 $x \in A$，存在惟一的 $y \in B$ 使得 $f(x,y)=0$，所以我们能够用条件 $g(x) \in B$ 和 $f(x,g(x))=0$ 定义一函数 $g: A \to \mathbf{R}$（如果 $b>0$，如图 2-4 所示，则 $g(x)=\sqrt{1-x^2}$）. 对我们所考虑的函数 f，还存在另一个数 b_1 使得 $f(a,b_1)=0$. 这时也会有包含 b_1 的一区间 B_1，使得当 $x \in A$ 时，有一个惟一的 $g_1(x) \in B_1$ 使 $f(x, g_1(x))=0$（这里 $g_1(x)=-\sqrt{1-x^2}$）. g 和 g_1 都可微. 这些函数叫做由方程 $f(x,y)=0$ **隐含地**定义的.

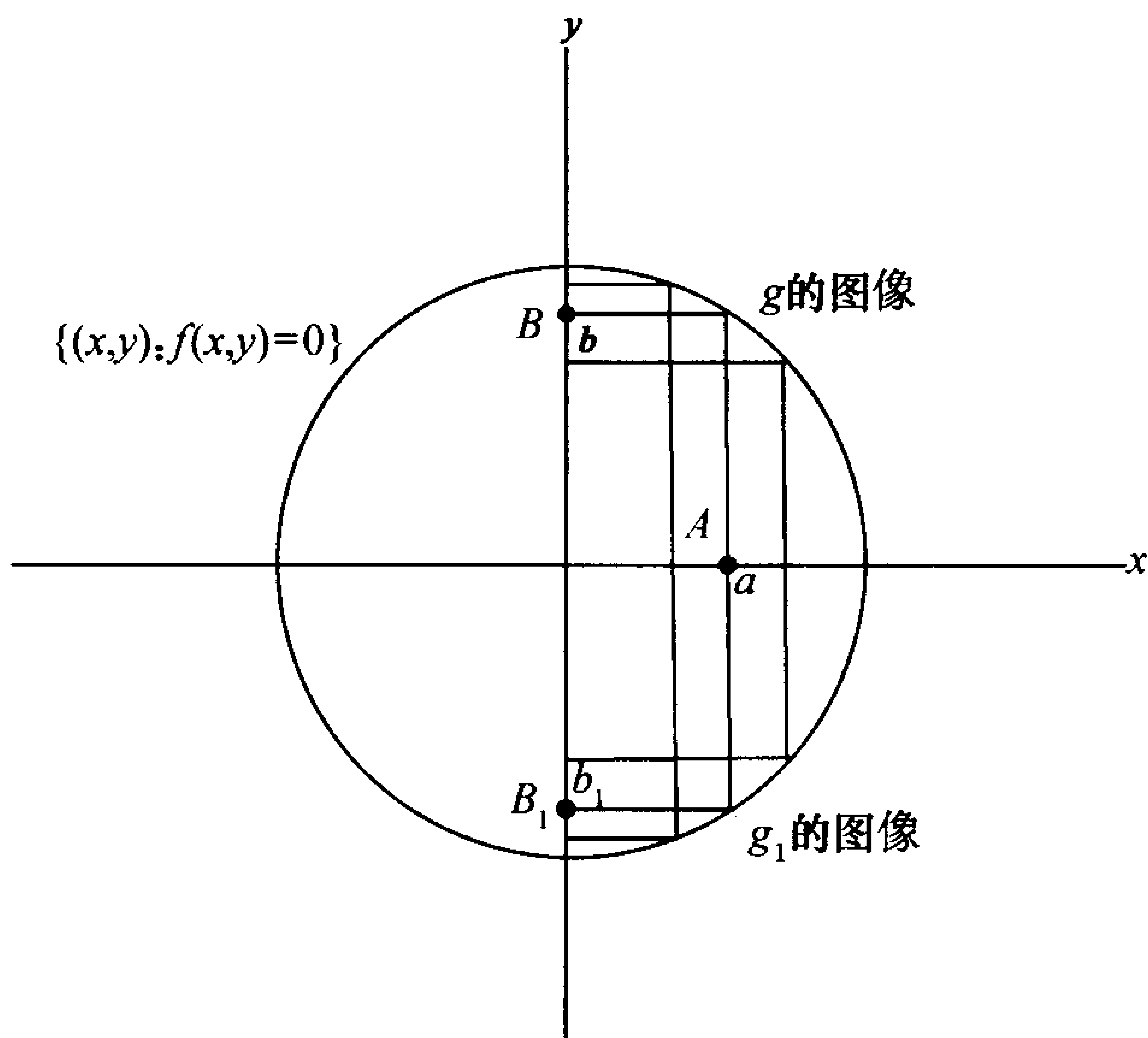


图 2-4

如果选 $a=1$ 或 -1，那么不可能找得定义在包含 a 的一个开区间中的任何一个这样的函数 g. 我们想要一个简单的判别法以决定一般在什么时候可以找得这样的函数. 更一般地，我们可以问：如 $f: \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ 且 $f(a^1,\cdots,a^n,b)=0$，对 $(a^1,\cdots,a^n)$ 附近的每个 $(x^1,\cdots,x^n)$，在什么情况下我们能找得 b 附近的惟一的 y 使得 $f(x^1,\cdots,x^n,y)=0$？甚至更为一般地，我们可以问是否可能求解依赖于参数 $x^1,\cdots,x^n$ 的含 m 个未知数的 m 个方程：若

$$f_i:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}\quad i=1,\cdots,m$$

且

$$f_i(a^1,\cdots,a^n,b^1,\cdots,b^m)=0\quad i=1,\cdots,m,$$

对 $(a^1,\cdots,a^n)$ 附近的每一 $(x^1,\cdots,x^n)$，在什么情况下，我们能够找得 $(b^1,\cdots,b^m)$ 附近的惟一的 $(y^1,\cdots,y^m)$，满足 $f_i(x^1,\cdots,x^n,y^1,\cdots,y^m)=0$？回答如下：

定理 2-12(隐函数定理) 设 $f:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}^m$ 在包含 (a,b) 的一开集中连续可微，且 $f(a,b)=0$. 令 M 表示 $m\times m$ 矩阵

$$(D_{n+j}f^i(a,b))\quad 1\leqslant i,j\leqslant m.$$

如 $\det M\neq0$，则必存在一个包含 a 的开集 $A\subset\mathbf{R}^n$ 和包含 b 的开集 $B\subset\mathbf{R}^m$，具有下列性质：对每一个 $x\in A$，存在惟一的 $g(x)\in B$ 使得 $f(x,g(x))=0$. 这个函数 g 还是可微的.

证 用 $F(x,y)=(x,f(x,y))$ 定义 $F:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}^n\times\mathbf{R}^m$，则 $\det F'(a,b)=\det M\neq0$. 则由定理 2-11，存在包含 $F(a,b)=(a,0)$ 的一开集 $W\subset\mathbf{R}^n\times\mathbf{R}^m$ 以及 $\mathbf{R}^n\times\mathbf{R}^m$ 中包含 (a,b) 的一个开集——我们可以把它取成 $A\times B$ 的形式使得 $F:A\times B\to W$ 有一个可微的逆 $h:W\to A\times B$. 显然 h 具有 $h(x,y)=(x,k(x,y))$ 的形式，其中 k 为某可微函数(因为 F 也有这种形式). 设 $\pi:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}^m$ 由 $\pi(x,y)=y$ 定义，则 $\pi\circ F=f$. 所以

$$\begin{aligned}f(x,k(x,y))&=f\circ h(x,y)=(\pi\circ F)\circ h(x,y)\\&=\pi\circ(F\circ h)(x,y)=\pi(x,y)=y.\end{aligned}$$

于是 $f(x,k(x,0))=0$；换句话说，我们可以定义 $g(x)=k(x,0)$. ▌

因为已知函数 g 是可微的，所以容易求出其导数. 事实上，由于 $f^i(x,g(x))=0$，两边同取 D_j 就给出

$$0=D_jf^i(x,g(x))+\sum_{\alpha=1}^{m}D_{n+\alpha}f^i(x,g(x))\cdot D_jg^\alpha(x)$$
$$i=1,\cdots,m;\quad j=1,\cdots,n.$$

因为 $\det M\neq0$，这些方程可对 $D_jg^\alpha(x)$ 求解. 解答将依赖于各个 D_jf

$(x, g(x))$，故也依赖于 $g(x)$. 这是不可避免的，因为函数 g 不是惟一的. 再次考察由 $f(x,y) = x^2 + y^2 - 1$ 定义的函数 $f: \mathbf{R}^2 \to \mathbf{R}$，注意到，满足 $f(x, g(x)) = 0$ 的两个可能的函数[1] 是 $g(x) = \sqrt{1 - x^2}$ 和 $g(x) = -\sqrt{1 - x^2}$. 将 $f(x, g(x)) = 0$ 求导，给出

$$D_1 f(x, g(x)) + D_2 f(x, g(x)) \cdot g'(x) = 0,$$

或即

$$2x + 2g(x) \cdot g'(x) = 0,$$
$$g'(x) = -x/g(x),$$

不论对 $g(x) = \sqrt{1 - x^2}$ 或 $g(x) = -\sqrt{1 - x^2}$ 确实都是如此.

可以给出定理 2-12 论证的一个推广，这在第 5 章中将是重要的.

定理 2-13　设 $f: \mathbf{R}^n \to \mathbf{R}^p$ 在包含 a 的一个开集中连续可微，其中 $p \leqslant n$. 如 $f(a) = 0$ 且 $p \times n$ 矩阵 $(D_j f^i(a))$ 有秩 p，则存在一开集 $A \subset \mathbf{R}^n$ 以及一个具可微逆的可微函数 $h: A \to \mathbf{R}^n$，使得

$$f \circ h(x^1, \cdots, x^n) = (x^{n-p+1}, \cdots, x^n).$$

证　我们可以把 f 看成是一个函数 $f: \mathbf{R}^{n-p} \times \mathbf{R}^p \to \mathbf{R}^p$. 若 $\det M \neq 0$，其中 M 是 $p \times p$ 矩阵 $(D_{n-p+j} f^i(a))$, $1 \leqslant i, j \leqslant p$，则这正好是定理 2-12 证明中所考虑的情况；正如在那个证明中指出的，存在 h 使得 $f \circ h(x^1, \cdots, x^n) = (x^{n-p+1}, \cdots, x^n)$.

一般地，因为 $(D_j f^i(a))$ 有秩 p，将存在 $j_1 < \cdots < j_p$ 使得矩阵 $(D_j f^i(a))$ $(1 \leqslant i \leqslant p, j = j_1, \cdots, j_p)$ 有非零的行列式. 若 $g: \mathbf{R}^n \to \mathbf{R}^n$ 置换各 x^j 使得 $g(x^1, \cdots, x^n) = (\cdots, x^{j_1}, \cdots, x^{j_p})$，则 $f \circ g$ 正是已经考察过的类型的函数，故对某个 k，$((f \circ g) \circ k)(x^1, \cdots, x^n) = (x^{n-p+1}, \cdots, x^n)$. 令 $h = g \circ k$. ▌

1. 当然指的是可微函数. ——译者注

习题

2-40. 利用隐函数定理重做习题2-15(c).

2-41. 设$f:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$是可微的. 对每一$x\in\mathbf{R}$用$g_x(y)=f(x,y)$定义$g_x:\mathbf{R}\to\mathbf{R}$. 假定对每一个$x$存在惟一的$y$使得$g'_x(y)=0$;令$c(x)$是这个$y$.

(a) 如对一切(x,y),$D_{2,2}f(x,y)\neq 0$, 试证c可微且

$$c'(x)=-\frac{D_{2,1}f(x,c(x))}{D_{2,2}f(x,c(x))}.$$

提示: $g'_x(y)=0$可以写成$D_2f(x,y)=0$.

(b) 试证: 如果$c'(x)=0$, 则对某一个y, 有

$$D_{2,1}f(x,y)=0,$$
$$D_2f(x,y)=0.$$

(c) 设$f(x,y)=x(y\ln y-y)-y\ln x$. 求

$$\max_{\frac{1}{2}\leqslant x\leqslant 2}\left(\min_{\frac{1}{3}\leqslant y\leqslant 1}f(x,y)\right).$$

2.7 记　　号

本节对与偏导数有联系的古典记号作一个简略的、但不是完全非主观的讨论. 热衷于古典记号的人们把偏导数$D_1f(x,y,z)$记作

$$\frac{\partial f(x,y,z)}{\partial x}\text{ 或 }\frac{\partial f}{\partial x}\text{ 或 }\frac{\partial f}{\partial x}(x,y,z)\text{ 或 }\frac{\partial}{\partial x}f(x,y,z)$$

或任何其他方便的类似记号. 这个记号迫使我们把$D_1f(u,v,w)$写成

$$\frac{\partial f}{\partial u}(u,v,w),$$

虽然可以用记号

$$\left.\frac{\partial f(x,y,z)}{\partial x}\right|_{(x,y,z)=(u,v,w)}\text{ 或 }\frac{\partial f(x,y,z)}{\partial x}(u,v,w)$$

或某种类似的东西(而对一个像$D_1f(7,3,2)$的式子就必须用这种记号). 对D_2f和D_3f也用了类似记号. 高阶导数用像这样的记号

$$\mathrm{D}_2\mathrm{D}_1 f(x,y,z) = \frac{\partial^2 f(x,y,z)}{\partial y \partial x}$$

来表示．当$f:\mathbf{R}\to\mathbf{R}$时，记号 ∂ 自动地恢复为 d；比如写作

$$\frac{\mathrm{d} \sin x}{\mathrm{d}x}\text{而不是}\frac{\partial \sin x}{\partial x}.$$

在古典记号下，仅就定理 2-2 的叙述而言就要引进一些不相干的字母．对 $\mathrm{D}_1(f\circ(g,h))$，通常的求法如下：

如$f(u,v)$ 是一函数，而 $u=g(x,y)$，$v=h(x,y)$，则

$$\frac{\partial f(g(x,y),h(x,y))}{\partial x} = \frac{\partial f(u,v)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f(u,v)}{\partial v}\frac{\partial v}{\partial x}.$$

[记号$\dfrac{\partial u}{\partial x}$表示$\dfrac{\partial}{\partial x}g(x,y)$，而$\dfrac{\partial}{\partial u}f(u,v)$ 表示 $\mathrm{D}_1 f(u,v) = \mathrm{D}_1 f(g(x,y),h(x,y))$.] 此式常简写成

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

注意，在这式子两边的f是有所区别的！

记号 $\mathrm{d}f/\mathrm{d}x$ 总多少有点儿诱人，它已分别引出许多关于 $\mathrm{d}x$ 和 $\mathrm{d}f$ 的定义(通常是无意义的)．其惟一的目的是得出式子

$$\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}x}\cdot \mathrm{d}x$$

若$f:\mathbf{R}^2\to\mathbf{R}$，则 $\mathrm{d}f$ 的古典定义为

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

（不管 $\mathrm{d}x$ 和 $\mathrm{d}y$ 表示什么）．

第 4 章包含一些严格的定义，这使得我们能够作为定理来证明以上各式．这些现代的定义是不是比古典的格式有实质性的进步，这是一个棘手的问题；读者必须自己来判定．

第3章 积　　分

3.1 基本定义

函数$f: A \to \mathbf{R}$的积分，这里$A \subset \mathbf{R}^n$是一个闭矩形，其定义和通常的积分如此相似，所以只须简短地讨论.

回想一下，闭区间$[a,b]$的分法P就是一串点$t_0,\cdots,t_k$，这里$a = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k = b$. 分法$P$把区间$[a,b]$分成$k$个子区间$[t_{i-1},t_i]$. 矩形$[a_1,b_1] \times \cdots \times [a_n,b_n]$的**分法$P$**就是一组分法$P = (P_1,\cdots,P_n)$,$P_i$是区间$[a_i,b_i]$的分法. 例如设$P_1 = t_0,\cdots,t_k$是$[a_1,b_1]$的一个分法，$P_2 = s_0,\cdots,s_l$是$[a_2,b_2]$的一个分法. 于是$[a_1,b_1] \times [a_2,b_2]$的分法$P = (P_1,P_2)$就把这个闭矩形分成$k \cdot l$个子矩阵，$[t_{i-1},t_i] \times [s_{j-1},s_j]$是其中典型的一个. 一般说来，如果$P_i$把$[a_i,b_i]$分成$N_i$个子区间，则$P = (P_1,\cdots,P_n)$把$[a_1,b_1] \times \cdots \times [a_n,b_n]$分成$N = N_1 \cdot \cdots \cdot N_n$个子矩形. 这些子矩形就叫做**分法$P$的子矩形**.

现在设A是一个矩形，$f: A \to \mathbf{R}$是一个有界函数，而P是A的一个分法. 对此分法的每个子矩形S，令

$$m_S(f) = \inf\{f(x): x \in S\},$$
$$M_S(f) = \sup\{f(x): x \in S\},$$

$v(S)$为S的体积［矩形$[a_1,b_1] \times \cdots \times [a_n,b_n]$和$(a_1,b_1) \times \cdots \times (a_n,b_n)$的**体积**都定义为$(b_1 - a_1)\cdots(b_n - a_n)$］. f关于P的**下和**和**上和**分别定义为

$$L(f,P) = \sum_S m_S(f) \cdot v(S), U(f,P) = \sum_S M_S(f) \cdot v(S).$$

显然$L(f,P) \leqslant U(f,P)$，还有更强的结论(3-2)也成立.

引理 3-1　设分法 P' 加细了 P（即 P' 的每个子矩形都包含在 P 的一个子矩形中）．这时

$$L(f,P) \leqslant L(f,P') \text{ 以及 } U(f,P') \leqslant U(f,P).$$

证　P 的每个子矩形 S 都被分成了 P' 的几个子矩形 $S_1,\cdots,S_\alpha$，于是 $v(S) = v(S_1) + \cdots + v(S_\alpha)$．现在 $m_S(f) \leqslant m_{S_i}(f)$，因为当 $x \in S$ 时的值 $f(x)$ 中包含了 $x \in S_i$ 时的全部值 $f(x)$（可能还包含有更小的值）．所以

$$\begin{aligned} m_S(f) \cdot v(S) &= m_S(f) \cdot v(S_1) + \cdots + m_S(f) \cdot v(S_\alpha) \\ &\leqslant m_{S_1}(f) \cdot v(S_1) + \cdots + m_{S_\alpha}(f) \cdot v(S_\alpha). \end{aligned}$$

左方对所有 S 各项之和是 $L(f,P)$，而右方各项之和是 $L(f,P')$．因此 $L(f,P) \leqslant L(f,P')$．对上和的证明类似．▌

推论 3-2　若 P 与 P' 是任意两个分法，则 $L(f,P') \leqslant U(f,P)$．

证　设分法 P'' 同时加细 P 和 P'（例如取 $P'' = (P''_1,\cdots,P''_n)$，$P''_i$ 是 $[a_i,b_i]$ 的一个同时加细 P_i 和 P'_i 的分法）．于是

$$L(f,P') \leqslant L(f,P'') \leqslant U(f,P'') \leqslant U(f,P).\ ▌$$

从推论 3-2 可得，f 之所有下和的上确界小于或等于 f 之所有上和的下确界．函数 $f:A\to\mathbf{R}$ 如果有界而且 $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ 就称 f 在矩形 A 上**可积**．上和下确界和下和上确界的公共值记作 $\int_A f$，并且称为 f 在 A 上的**积分**．时常也采用 $\int_A f(x^1,\cdots,x^n)\mathrm{d}x^1\cdots\mathrm{d}x^n$ 这样的记号．如果 $f:[a,b]\to\mathbf{R}$，而且 $a\leqslant b$，则 $\int_a^b f = \int_{[a,b]} f$．下面的定理给出了可积性的一个简单而有用的判别法．

定理 3-3　有界函数 $f:A\to\mathbf{R}$ 为可积的充分必要条件是对任一 $\varepsilon>0$ 都有 A 的一个分法 P，使得 $U(f,P) - L(f,P) < \varepsilon$．

证　若此条件成立，显然 $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ 且 f 可积．另一方面，若 f 可积，则 $\sup\{L(f,P)\} = \inf\{U(f,P)\}$，于是对任一 $\varepsilon>0$ 必定有分法 P 和 P' 存在使 $U(f,P) - L(f,P') < \varepsilon$．若 P'' 同

时加细 P 和 P'', 由引理3-1 有 $U(f,P'') - L(f,P'') \leqslant U(f,P) - L(f,P') < \varepsilon$. ▌

我们将在后面各节里刻画可积函数的特性并且找出计算积分的一种方法. 目前我们只考虑两个函数，一个可积，一个不可积.

1. 令 $f: A \to \mathbf{R}$ 是常值函数, $f(x) = c$. 于是对任意分法 P 和子矩形 S 都有 $m_S(f) = M_S(f) = c$, 因此 $L(f,P) = U(f,P) = \sum_S c \cdot v(S) = c \cdot v(A)$. 所以 $\int_A f = c \cdot v(A)$.

2. 令 $f: [0,1] \times [0,1] \to \mathbf{R}$ 定义为

$$f(x,y) = \begin{cases} 0 & \text{若 } x \text{ 是有理数}, \\ 1 & \text{若 } x \text{ 是无理数}. \end{cases}$$

若 P 是一个分法，则任一个子矩形 S 既包含 x 为有理数的点 (x,y)，也包含 x 为无理数的点 (x,y). 所以 $m_S(f) = 0$ 而 $M_S(f) = 1$. 于是

$$L(f,P) = \sum_S 0 \cdot v(S) = 0$$

而

$$U(f,P) = \sum_S 1 \cdot v(S) = v([0,1] \times [0,1]) = 1.$$

所以 f 不可积.

习题

3-1. 令 $f: [0,1] \times [0,1] \to \mathbf{R}$ 定义为

$$f(x,y) = \begin{cases} 0 & \text{若 } 0 \leqslant x < \dfrac{1}{2}, \\ 1 & \text{若 } \dfrac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

证明 f 可积而且 $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

3-2. 令 $f: A \to \mathbf{R}$ 可积而除在有限多个点以外 $g = f$. 证明 g 是可积的而且 $\int_A f = \int_A g$.

3-3. 令$f, g: A \to \mathbf{R}$均为可积.

(a) 对A的任意分法P及其子矩形S，求证$m_S(f)+m_S(g) \leqslant m_S(f+g)$，而$M_S(f+g) \leqslant M_S(f)+M_S(g)$，因此$L(f,P)+L(g,P) \leqslant L(f+g,P)$，而$U(f+g,P) \leqslant U(f,P)+U(g,P)$.

(b) 证明$f+g$也可积而且$\int_A (f+g) = \int_A f + \int_A g$.

(c) 对任意常数c，证明$\int_A cf = c\int_A f$.

3-4. 令$f: A \to \mathbf{R}$，而P是A的一个分法. 证明f为可积当且仅当对于A的每个子矩形S，函数$f|S$(即f限制在S上)均为可积，并证这时$\int_A f = \sum_S \int_S f|S$.

3-5. 令$f, g: A \to \mathbf{R}$均为可积并设$f \leqslant g$. 求证$\int_A f \leqslant \int_A g$.

3-6. 令$f: A \to \mathbf{R}$为可积，证明$|f|$可积而且$\left|\int_A f\right| \leqslant \int_A |f|$.

3-7. 令$f: [0,1] \times [0,1] \to \mathbf{R}$定义为

$$f(x,y) = \begin{cases} 0 & x \text{ 为无理数}, \\ 0 & x \text{ 为有理数}, y \text{ 为无理数} \\ 1/q & x \text{ 为有理数}, y \text{ 为既约分数 } p/q, p, q \text{ 均} > 0. \end{cases}$$

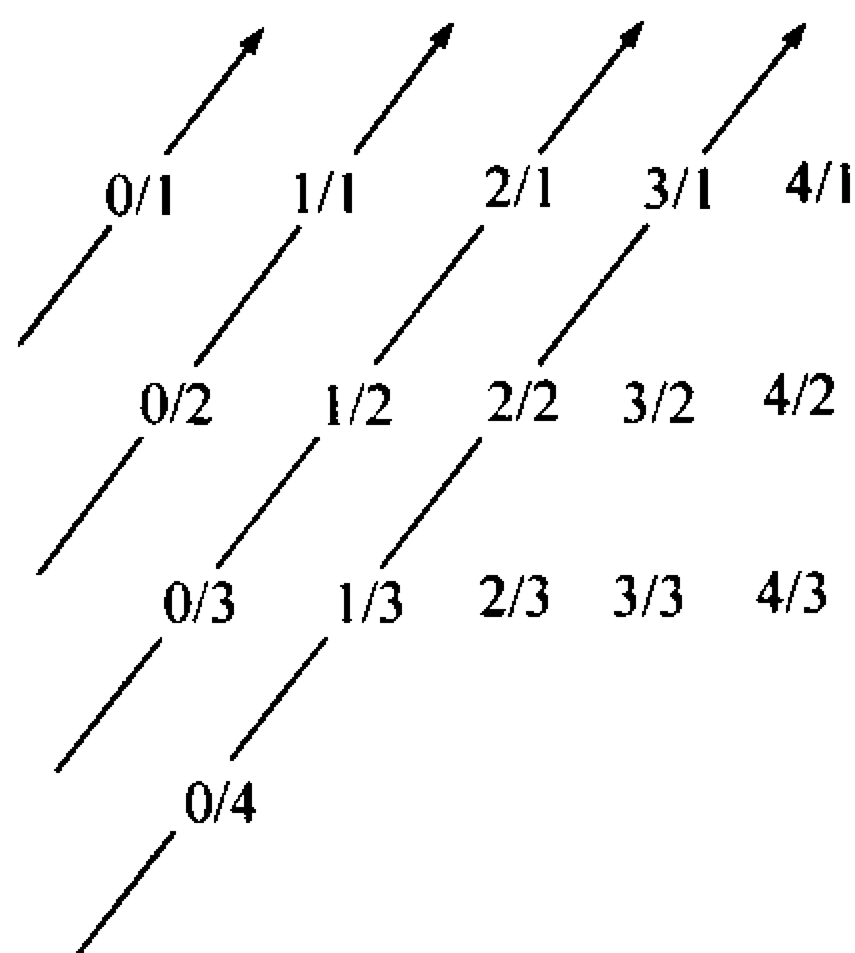
证明f可积而且$\int_{[0,1]\times[0,1]} f = 0$.

3.2 测度零与容度零

$\mathbf{R}^n$的子集A具有(n维)测度0，如果对任何$\varepsilon > 0$都有A的闭矩形的覆盖$\{U_1, U_2, U_3, \cdots\}$使得$\sum_{i=1}^{\infty} v(U_i) < \varepsilon$. 很明显，若$A$有测度0而且$B \subset A$，则$B$也有测度0（记住这一点是有用的）. 读者可以验证，在测度0的定义中可以用开矩形代替闭矩形.

只含有限多个点的集明显地测度为0. 若A含有无限多个点但是可以排成一序列$a_1, a_2, a_3, \cdots$，则A也有测度0. 因为若$\varepsilon > 0$，我们可取闭矩形U_i包含a_i，而且$v(U_i) < \varepsilon/2^i$. 这时$\sum_{i=1}^{\infty} v(U_i) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$.

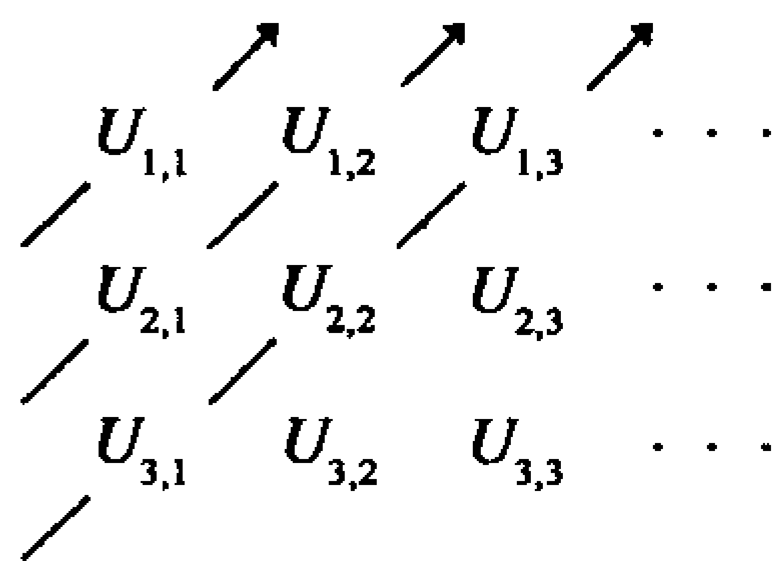
0 与 1 之间的全体有理数之集是可以排成一可数的无限集，这是一个重要而且令人惊讶的例子. 为了看出这一点，把下面的阵列中的分数按箭头次序排列起来(去掉重复的和大于 1 的数)：



可以给出这个思想的一个重要推广.

定理 3-4 若 $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ 而每个 A_i 均有测度 0，则 A 也有测度 0.

证 令 $\varepsilon > 0$. 因为 A_i 有测度 0，故有 A_i 的闭矩形覆盖 $\{U_{i,1}, U_{i,2}, U_{i,3}, \cdots\}$ 使 $\sum_{j=1}^{\infty} v(U_{i,j}) < \varepsilon/2^i$. 于是全体 $U_{i,j}$ 的族形成 A 的一个覆盖. 考虑阵列



我们看到，这个族可以排成一序列 $V_1, V_2, V_3, \cdots$，很明显 $\sum_{i=1}^{\infty} v(V_i) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$. ▌

$\mathbf{R}^n$ 的子集 A，如果对每个 $\varepsilon > 0$ 都有闭矩形构成的 A 的有限覆盖 $\{U_1, \cdots, U_n\}$ 使 $\sum_{i=1}^{n} v(U_i) < \varepsilon$，就说是具有($n$ 维)**容度** 0. 如果 A 具有容度 0，很明显它也具有测度 0. 定义中的闭矩形还是可以换成开矩形.

定理 3-5 若 $a<b$，则 $[a,b]\subset\mathbf{R}$ 不能有容度0. 事实上，若 $\{U_1,\cdots,U_n\}$ 是 $[a,b]$ 的闭区间有限覆盖，则 $\sum_{i=1}^{n}v(U_i)\geqslant b-a$.

证 很明显可以假设每个 $U_i\cap[a,b]\neq\varnothing$. 令 $a=t_0<t_1<\cdots<t_k=b$ 是所有 U_i 的所有端点. 于是每个 $v(U_i)$ 是某些 t_j-t_{j-1} 的和. 此外，每个 $[t_{j-1},t_j]$ 至少含于一个 U_i 之内（即任一包含 $[t_{j-1},t_j]$ 的内点的 U_i），所以

$$\sum_{i=1}^{n}v(U_i)\geqslant\sum_{j=1}^{k}(t_j-t_{j-1})=b-a. \quad \blacksquare$$

若 $a<b,[a,b]$ 也不会有测度0，这是由于

定理 3-6 若 A 为紧的且有测度0，则 A 也有容度0.

证 令 $\varepsilon>0$. 因为 A 有测度0，所以 A 有开矩形的覆盖 $U_1,U_2,\cdots$ 而且 $\sum_{i=1}^{\infty}v(U_i)<\varepsilon$. 由于 A 为紧的，这些 U_i 中的有限个 $U_1,\cdots,U_n$ 即可覆盖 A，而且必定有 $\sum_{i=1}^{n}v(U_i)<\varepsilon$. $\blacksquare$

若 A 非紧，定理3-6的结论不真. 例如，令 A 为0和1之间的有理数集，则 A 有测度0. 但若 $\{[a_1,b_1],\cdots,[a_n,b_n]\}$ 覆盖 A. 则 A 必包含于闭集 $[a_1,b_1]\cup\cdots\cup[a_n,b_n]$ 中，从而 $[0,1]\subset[a_1,b_1]\cup\cdots\cup[a_n,b_n]$. 由定理3-5知对任意这样的覆盖 $\sum_{i=1}^{n}(b_i-a_i)\geqslant1$，所以 A 不能有容度0.

习题

3-8. 证明：若对每一 i 如果 $a_i<b_i$，则 $[a_1,b_1]\times\cdots\times[a_n,b_n]$ 不能有容度0.

3-9. (a) 证明无界集不能有容度0.

(b) 作一个测度为0而容度不为0的闭集的例子.

3-10. (a) 若 C 是具有容度0的集，证明 C 的边界也有容度0.

(b) 作出一个测度为0的有界集 C 的例子，使 C 的边界测度不为0.

3-11. 令 A 为习题1-18中的集. 若 $\sum_{i=1}^{\infty}(b_i-a_i)<1$，证明 A 的边界不能有测度0.

3-12. 令 $f:[a,b]\to\mathbf{R}$ 是一个增函数. 证明 $\{x: f$在x点不连续$\}$ 有测度0. 提示：用习题1-30证明 $\{x:o(f,x) > 1/n\}$ 对每个整数 n 都是有限集.

3-13. *(a) 证明一切矩形 $[a_1,b_1]\times\cdots\times[a_n,b_n]$ 之集可排成一序列，其中所有 a_i 和 b_i 均为有理数.

(b) 若 $A\subset\mathbf{R}^n$ 是任一集，$\mathcal{O}$ 是 A 的开覆盖，证明必存在 $\mathcal{O}$ 中之元素的序列 $U_1,U_2,U_3,\cdots$ 也覆盖 A. 提示：对于每一点 $x\in A$，都有一个矩形 $B=[a_1,b_1]\times\cdots\times[a_n,b_n]$，其中 a_i，b_i 都是有理数，使得 $x\in B\subset U$，$U\in\mathcal{O}$.

3.3 可积函数

回想一下，$o(f,x)$表示f在x处的振幅.

引理3-7 令A为闭矩形而$f:A\to\mathbf{R}$是一个有界函数且对一切$x\in A$都有$o(f,x)<\varepsilon$. 这时必有A的一个分法P使$U(f,P)-L(f,P)<\varepsilon\cdot v(A)$.

证 对每一点$x\in A$都有一个闭矩形U_x以x为其内点，使得$M_{U_x}(f)-m_{U_x}(f)<\varepsilon$. 既然$A$为紧集，集$U_x$中有限多个$U_{x_1},\cdots,U_{x_n}$即可覆盖$A$. 令$P$是$A$的一个分法使其每一个子矩形$S$都含在某个$U_{x_i}$中. 这时$M_S(f)-m_S(f)<\varepsilon$对于$P$的每一个子矩形$S$都成立，所以

$$\begin{aligned}U(f,P)-L(f,P)&=\sum_S[M_S(f)-m_S(f)]\cdot v(S)\\&<\varepsilon\cdot v(A).\end{aligned}$$ ▌

定理3-8 令A为闭矩形而$f:A\to\mathbf{R}$是有界函数. 令$B=\{x:f$在x点不连续$\}$. 当且仅当B为测度0集时f为可积.

证 先设B有测度0. 令$\varepsilon>0$，$B_\varepsilon=\{x:o(f,x)\geqslant\varepsilon\}$. 于是$B_\varepsilon\subset B$，而$B_\varepsilon$有测度0. 因为$B_\varepsilon$为紧集（定理1-11），故$B_\varepsilon$有容度0. 这样必有闭矩形的有限族$U_1,\cdots,U_n$其内域覆盖$B_\varepsilon$而且$\sum_{i=1}^{n}v(U_i)<\varepsilon$. 令$P$为$A$的分法且使其每一个$P$的子矩形都分属以下两组（见

图3-1)：

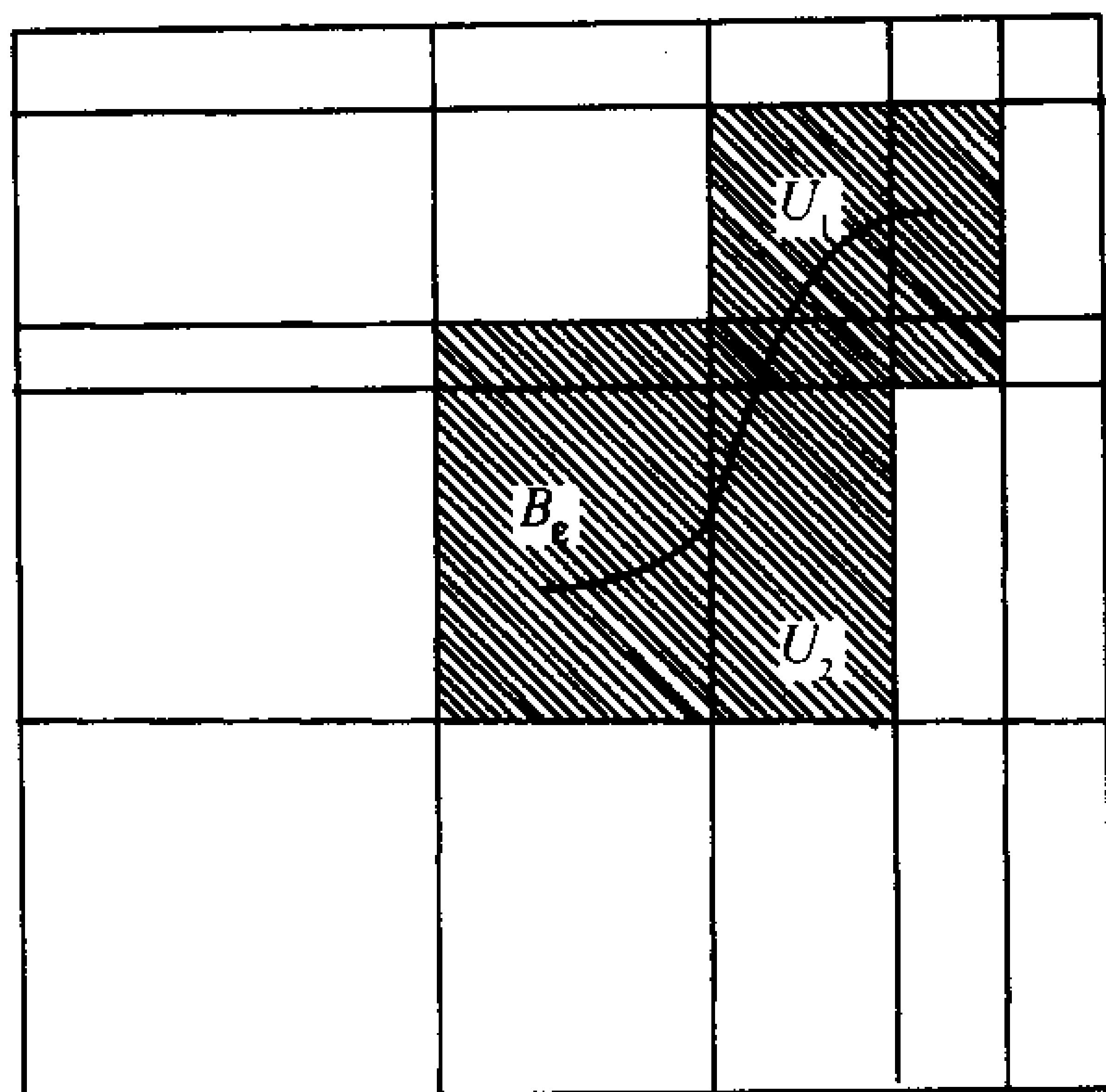


图3-1　有阴影的矩形在δ_1中

(1) $\mathcal{S}_1$，它包括含于某个U_i内的子矩形S:$S \subset U_i$.

(2) $\mathcal{S}_2$，它包括能使$S \cap B_\varepsilon = \varnothing$的子矩形$S$.

设当$x \in A$时$|f(x)| < M$. 于是对每个S, $M_S(f) - m_S(f) < 2M$. 因此

$$\sum_{S \in \mathcal{S}_1} [M_S(f) - m_S(f)] \cdot v(S) < 2M \sum_{i=1}^{n} v(U_i) < 2M\varepsilon .$$

若$S \in \mathcal{S}_2$，则对$x \in S$有$o(f,x) < \varepsilon$. 引理3-7导出，存在着P的一个细分P'使得对于$S \in \mathcal{S}_2$有

$$\sum_{S' \subset S} [M_{S'}(f) - m_{S'}(f)] \cdot v(S') < \varepsilon \cdot v(S).$$

于是

$$\begin{aligned} U(f,P') - L(f,P') &= \sum_{S' \subset S \in \mathcal{S}_1} [M_{S'}(f) - m_{S'}(f)] \cdot v(S') \\ &\quad + \sum_{S' \subset S \in \mathcal{S}_2} [M_{S'}(f) - m_S{}'(f)] \cdot v(S') \\ &< 2M\varepsilon + \sum_{S \in \mathcal{S}_2} \varepsilon \cdot v(S) \\ &\leqslant 2M\varepsilon + \varepsilon \cdot v(A). \end{aligned}$$

因为 M 和 $v(A)$ 都固定，上式说明可以找到一个分法 P' 使 $U(f,P')-L(f,P')$ 可任意小．故 f 可积．

反过来，设 f 可积，因为 $B=B_1\cup B_{\frac{1}{2}}\cup B_{\frac{1}{3}}\cup\cdots$，所以(由定理 3-4)只要证明每一个 $B_{\frac{1}{n}}$ 都有测度 0 即可．事实上我们将证明每个 $B_{\frac{1}{n}}$ 均有容度 0(因 $B_{\frac{1}{n}}$ 是紧的，这与测度 0 是等价的)．

若 $\varepsilon>0$，令 P 是 A 的一个分法且使 $U(f,P)-L(f,P)<\varepsilon/n$．令 $\mathcal{S}$ 为 P 中与 $B_{\frac{1}{n}}$ 相交的 S 之集．于是 $\mathcal{S}$ 是 $B_{\frac{1}{n}}$ 的覆盖．如果 $S\in\mathcal{S}$，则 $M_S(f)-m_S(f)\geqslant\frac{1}{n}$．于是

$$\begin{aligned}\frac{1}{n}\cdot\sum_{S\in\mathcal{S}}v(S)&\leqslant\sum_{S\in\mathcal{S}}[M_S(f)-m_S(f)]\cdot v(S)\\&\leqslant\sum_{S}[M_S(f)-m_S(f)]\cdot v(S)<\frac{\varepsilon}{n},\end{aligned}$$

从而 $\sum_{S\in\mathcal{S}}v(S)<\varepsilon$． ▌

至今我们只讨论过函数在矩形上的积分．在其他集上的积分很容易划归这种类型．若 $C\subset\mathbf{R}^n$，定义 C 的**特征函数** χ_C 为

$$\chi_C(x)=\begin{cases}0 & x\notin C,\\1 & x\in C.\end{cases}$$

若对某个闭矩形 A 有 $C\subset A$，而 $f:A\to\mathbf{R}$ 为有界，只要 $f\cdot\chi_C$ 是可积的，则定义 $\int_C f$ 为 $\int_A f\cdot\chi_C$．如果 f 和 χ_C 都是可积的，那么 $f\cdot\chi_C$ 一定是可积的(习题 3-14)．

定理 3-9 函数 $\chi_C:A\to\mathbf{R}$ 当且仅当 C 的边界具有测度 0 (从而也具有容度 0) 时可积．

证 若 x 在 C 的内域中，则有一个开矩形 U 使 $x\in U\subset C$．于是在 U 上 $\chi_C=1$ 而且 χ_C 在 x 点显然连续．同样，若 x 在 C 的外域中，也有一个开矩形 U 使 $x\in U\subset\mathbf{R}^n-C$．于是在 U 上 $\chi_C=0$ 而 χ_C 在 x 点连续．最后，若 x 在 C 的边界上，则对任一包含 x 点的开矩形 U，

其中有 $y_1 \in U \cap C$ 从而 $\chi_C(y_1)=1$，也有 $y_2 \in U \cap (\mathbf{R}^n - C)$ 从而 $\chi_C(y_2)=0$. 因此 χ_C 在 x 不连续. 于是 $\{x: \chi_C$ 在 x 点不连续$\} = (C$ 的边界)，再由定理 3-8 即得结论. ▌

边界具有测度 0 的有界集称为是**约当**（Jordan）**可测**的. 积分 $\int_C 1$ 称为 C 的(n 维)**容度**或(n 维)**体积**. 一维体积自然地常称为**长度**，二维体积称为**面积**.

习题 3-11 说明甚至开集 C 也可能不是约当可测的，于是即使令 C 是开集而 f 连续，$\int_C f$ 也不一定有定义. 这种令人不快的情况马上就要加以纠正.

习题

3-14. 证明若 $f, g: A \to \mathbf{R}$ 均为可积，则 $f \cdot g$ 也可积.

3-15. 证明若 C 具有容度 0，则必有某个闭矩形 A 使 $C \subset A$ 成立，而且 C 是约当可测的，以及 $\int_A \chi_C = 0$.

3-16. 作一个具有测度 0 的有界集 C 的例子使得 $\int_A \chi_C$ 不存在.

3-17. 若 C 是一个测度 0 的有界集而且 $\int_A \chi_C$ 存在，求证 $\int_A \chi_C = 0$. 提示：证明对一切分法 P，$L(\chi_C, P) = 0$. 用习题 3-8.

3-18. 若 $f: A \to \mathbf{R}$ 是非负的，而且 $\int_A f = 0$，求证 $\{x: f(x) \neq 0\}$ 具有测度 0. 提示：求证对一切正整数 m 有 $\left\{x: f(x) > \frac{1}{m}\right\}$ 有容度 0.

3-19. 令 U 为习题 3-11 中的开集，若除一个测度 0 的集外有 $f = \chi_C$，则 f 在 $[0,1]$ 上不可积.

3-20. 证明增函数 $f: [a,b] \to \mathbf{R}$ 在 $[a,b]$ 上可积.

3-21. 若 A 为闭矩形，求证 $C \subset A$ 为约当可测当且仅当对任意 $\varepsilon > 0$ 必有 A 的一个分法 P 使得 $\sum_{S \in \mathcal{S}_1} v(S) - \sum_{S \in \mathcal{S}_2} v(S) < \varepsilon$，$\mathcal{S}_1$ 表示一切与 C 相交的子矩形之集，$\mathcal{S}_2$ 表示一切含于 C 内的子矩形的集.

3-22. *若 A 为约当可测集且 $\varepsilon > 0$，求证必有一个紧约当可测集 $C \subset A$ 使得 $\int_{A-C} 1 < \varepsilon$.

3.4 富比尼定理

定理 3-10 在某种意义下解决了积分的计算问题，它把 $\mathbf{R}^n$ $(n>1)$ 中闭矩形上积分的计算化为 $\mathbf{R}$ 中闭区间上积分的计算. 这个定理很重要，值得专门起个名字，通常称为富比尼(Fubini)定理，虽然它只是富比尼所证明的一个定理的特例，而且是在定理 3-10 发现多年后，才被富比尼证明的.

这个定理蕴涵的想法最好是用正连续函数 $f:[a,b]\times[c,d]\to\mathbf{R}$ 来说明(图 3-2). 令 $t_0,\cdots,t_n$ 是 $[a,b]$ 的一个分法，并用线段 $\{t_i\}\times[c,d]$ 把 $[a,b]\times[c,d]$ 分成 n 条. 若用 $g_x(y)=f(x,y)$ 定义 g_x，则在 f 之图像下方以及 $\{x\}\times[c,d]$ 上方所围的区域的面积是

$$\int_c^d g_x = \int_c^d f(x,y)\,\mathrm{d}y.$$

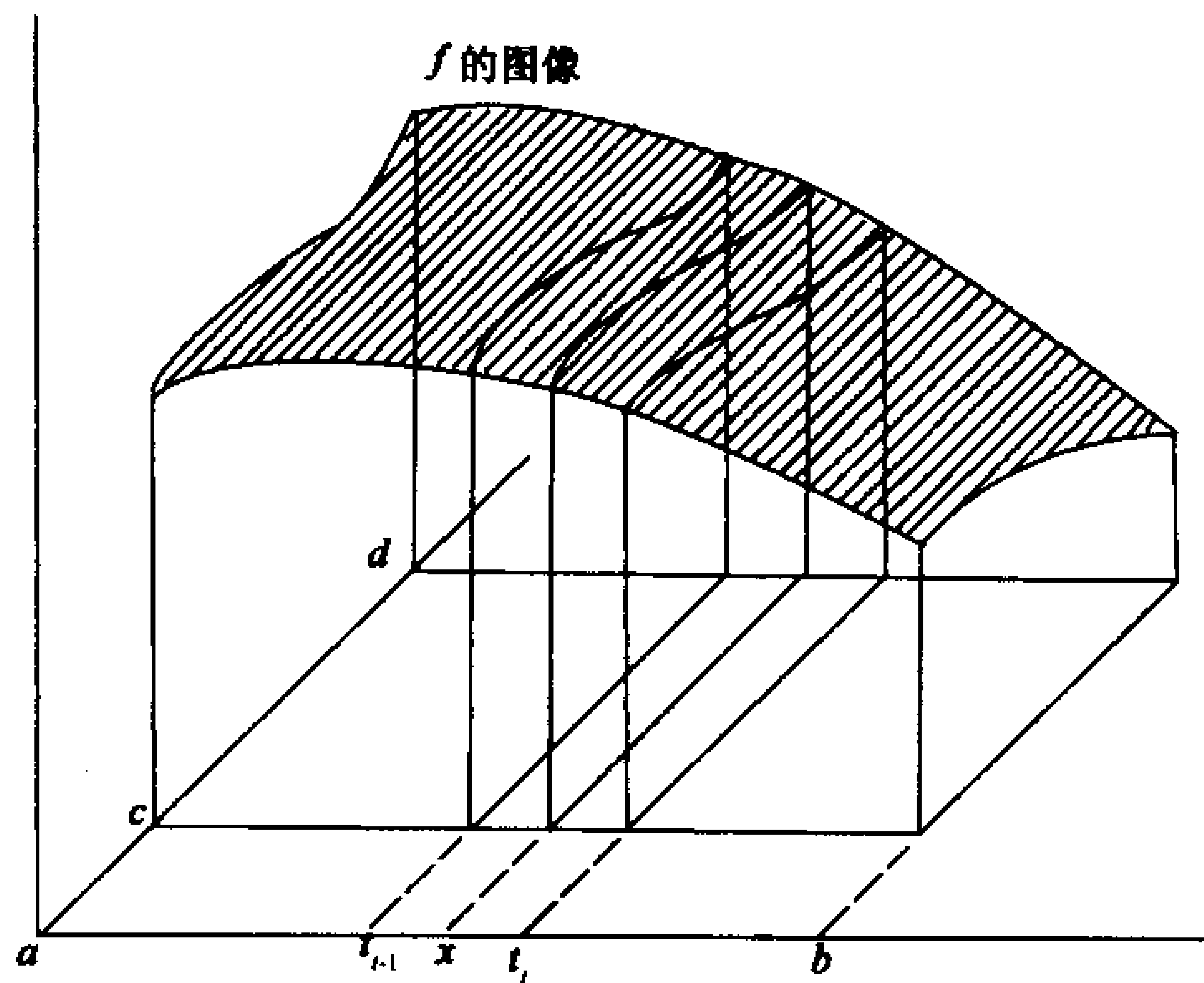


图 3-2

所以在 f 图像下方以及 $[t_{i-1},t_i]\times[c,d]$ 上方的区域的体积近似

等于 $(t_i - t_{i-1}) \cdot \int_c^d f(x,y)\mathrm{d}y$，对任意 $x \in [t_{i-1}, t_i]$. 故

$$\int_{[a,b]\times[c,d]} f = \sum_{i=1}^{n} \int_{[t_{i-1},t_i]\times[c,d]} f$$

之值近似等于 $\sum_{i=1}^{n} (t_i - t_{i-1}) \cdot \int_c^d f(x_i, y)\mathrm{d}y$，$x_i$ 在 $[t_{i-1}, t_i]$ 之中. 另一方面，类似的和也在 $\int_a^b \left(\int_c^d f(x,y)\mathrm{d}y\right)\mathrm{d}x$ 的定义中出现. 所以，若 h 是由 $h(x) = \int_c^d g_x = \int_c^d f(x,y)\mathrm{d}y$ 定义的，就有理由设想 h 在 $[a,b]$ 上可积，而且

$$\int_{[a,b]\times[c,d]} f = \int_a^b h = \int_a^b \left(\int_c^d f(x,y)\mathrm{d}y\right)\mathrm{d}x.$$

事实上，当 f 为连续时这确实是成立的. 但在一般情况下可能存在问题. 例如，设 f 的不连续点集是 $\{x_0\} \times [c,\mathrm{d}]$，$x_0$ 是 $[a,b]$ 中某点. 于是 f 在 $[a,b] \times [c,\mathrm{d}]$ 上可积，但 $h(x_0) = \int_c^d f(x_0,y)\mathrm{d}y$ 甚至无定义. 所以富比尼定理的提法看起来有点奇特，在它后面将要加一些关于各种特殊情形的注解，这时的提法可能比较简单.

我们要用到一些名词. 若 $f: A \to \mathbf{R}$ 是一个闭矩形上的有界函数，这时，不论 f 是否可积，所有下和的上确界和所有上和的下确界都存在. 它们分别称为 f 在 A 上的**下积分**和**上积分**，并且记作

$$\mathbf{L}\int_A f \text{ 和 } \mathbf{U}\int_A f,$$

定理 3-10（富比尼定理）　令 $A \subset \mathbf{R}^n$，$B \subset \mathbf{R}^m$ 均为闭矩形，$f: A \times B \to \mathbf{R}$ 为可积. 对 $x \in A$，定义 $g_x: B \to \mathbf{R}$ 为 $g_x(y) = f(x,y)$，再令

$$\mathscr{L}(x) = \mathbf{L}\int_B g_x = \mathbf{L}\int_B f(x,y)\mathrm{d}y,$$

$$\mathscr{U}(x) = \mathbf{U}\int_B g_x = \mathbf{U}\int_B f(x,y)\mathrm{d}y.$$

于是 $\mathscr{L}$ 和 $\mathscr{U}$ 在 A 上均可积，而且

$$\int_{A\times B} f = \int_A \mathfrak{L} = \int_A \left(\mathbf{L}\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x,$$

$$\int_{A\times B} f = \int_A \mathscr{U} = \int_A \left(\mathbf{U}\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x,$$

(式子右方的积分称为f的**逐次积分**.)

证 令P_A是A的一个分法，P_B是B的一个分法. 它们可以合并成$A\times B$的一个分法P，其子矩形S都是$S_A\times S_B$这样的形状，其中S_A是分法P_A的子矩形，S_B是分法P_B的子矩形. 于是

$$\begin{aligned} L(f,P) &= \sum_S m_S(f)\cdot v(S) = \sum_{S_A,S_B} m_{S_A\times S_B}(f)\cdot v(S_A\times S_B) \\ &= \sum_{S_A}\Big(\sum_{S_B} m_{S_A\times S_B}(f)\cdot v(S_B)\Big)\cdot v(S_A). \end{aligned}$$

现在，如果$x\in S_A$，那么显然有$m_{S_A\times S_B}(f)\leqslant m_{S_B}(g_x)$. 从而对$x\in S_A$有

$$\begin{aligned} \sum_{S_B} m_{S_A\times S_B}(f)\cdot v(S_B) &\leqslant \sum_{S_B} m_{S_B}(g_x)\cdot v(S_B) \\ &\leqslant \mathbf{L}\int_B g_x = \mathfrak{L}(x) \end{aligned}$$

所以

$$\sum_{S_A}\Big(\sum_{S_B} m_{S_A\times S_B}(f)\cdot v(S_B)\Big)\cdot v(S_A) \leqslant L(\mathfrak{L},P_A)$$

于是我们得到

$$L(f,P)\leqslant L(\mathfrak{L},P_A)\leqslant U(\mathfrak{L},P_A)\leqslant U(\mathscr{U},P_A)\leqslant U(f,P),$$

最后一个不等式的证法和第一个的证法完全一样. 因为f可积，$\sup\{L(f,P)\} = \inf\{U(f,P)\} = \int_{A\times B} f$. 于是

$$\sup\{L(\mathfrak{L},P_A)\} = \inf\{U(\mathfrak{L},P_A)\} = \int_{A\times B} f.$$

换言之，$\mathfrak{L}$在A上可积而且$\int_{A\times B} f = \int_A \mathfrak{L}$. 从不等式

$$L(f,P) \leqslant L(\mathfrak{L},P_A) \leqslant L(\mathscr{U},P_A) \leqslant U(\mathscr{U},P_A) \leqslant U(f,P)$$

可以类似地证明关于$\mathscr{U}$的结论. ▌

注　1. 用类似的证法可证

$$\int_{A\times B} f = \int_B \left(\mathbf{L}\int_A f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_B \left(\mathbf{U}\int_A f(x,y)\,\mathrm{d}x\right)\mathrm{d}y$$

这些积分叫做与定理中次序相反的f的逐次积分. 后面几个习题表明，逐次积分交换次序的可能性有许多推论.

2. 实际上时常出现每一个g_x都可积的情况，这时

$$\int_{A\times B} f = \int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

当f连续时，上式一定成立.

3. 时常遇到的最坏的不正规性的情况是g_x只对有限多个$x \in A$不可积. 这时，除有限多个x外，$\mathfrak{L}(x) = \int_B f(x,y)\,\mathrm{d}y$. 因为当在有限多个点上重新规定$\mathfrak{L}$之值时，$\int_A \mathfrak{L}$之值不会改变，所以若在$\int_B f(x,y)\,\mathrm{d}y$不存在时任意给它一个值，例如令它为0，我们仍可写成$\int_{A\times B} f = \int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$.

4. 在有些情况下这也行不通，定理3-10必须按照上面的提法来用. 令$f:[0,1]\times[0,1]\to\mathbf{R}$定义为

$$f(x,y) = \begin{cases} 1 & \text{若 } x \text{ 为无理数}, \\ 1 & \text{若 } x \text{ 为有理数而 } y \text{ 为无理数}, \\ 1-1/q & \text{若 } x=p/q\ (q>0) \text{ 为既约分数而 } y \text{ 为有理数}. \end{cases}$$

这时f可积且$\int_{[0,1]\times[0,1]} f = 1$. 若$x$为无理数，$\int_0^1 f(x,y)\,\mathrm{d}y = 1$，而当$x$为有理数，它则不存在. 所以，当积分$h(x) = \int_0^1 f(x,y)\,\mathrm{d}y$不存在时令它为0，则$h$不可积.

5. 若$A = [a_1,b_1]\times\cdots\times[a_n,b_n]$而$f:A\to\mathbf{R}$可积，我们就可以

反复应用富比尼定理而得

$$\int_A f = \int_{a_n}^{b_n}\left(\cdots\int_{a_1}^{b_1} f(x^1,\cdots,x^n)\,dx^1)\cdots\right)dx^n.$$

6. 若 $C \subset A \times B$, 富比尼定理也能用来计算 $\int_C f$, 因为它是由 $\int_{A\times B}\chi_C f$ 定义的. 例如,设

$$C = [-1,1]\times[-1,1] - \{(x,y): |(x,y)| < 1\}.$$

则

$$\int_C f = \int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)\cdot\chi_C(x,y)\,dy\right)dx.$$

现在

$$\chi_C(x,y) = \begin{cases} 1 & \text{若 } y > \sqrt{1-x^2} \text{ 或 } y < -\sqrt{1-x^2}, \\ 0 & \text{其他情况.} \end{cases}$$

所以

$$\int_{-1}^{1} f(x,y)\cdot\chi_C(x,y)\,dy = \int_{-1}^{-\sqrt{1-x^2}} f(x,y)\,dy + \int_{\sqrt{1-x^2}}^{1} f(x,y)\,dy.$$

一般来说，若 $C \subset A \times B$，在推导 $\int_C f$ 的表达式过程中，主要困难在于决定 $C \cap (\{x\}\times B), x \in A$. 对 $y \in B, C \cap (A \times \{y\})$ 如果是比较容易决定的话，那么就应该应用逐次积分

$$\int_C f = \int_B\left(\int_A f(x,y)\cdot\chi_C(x,y)\,dx\right)dy.$$

习题

3-23. 令 $C \subset A\times B$ 具有容度 0. 令 $A' \subset A$ 是使 $\{y \in B: (x,y) \in C\}$ 不具有容度 0 的一切 $x \in A$ 之集. 证明 A' 是一个测度为 0 的集. 提示: χ_C 可积而 $\int_{A\times B}\chi_C$

$= \int_A \mathscr{U} = \int_A \mathfrak{L}$,所以$\int_A \mathscr{U} - \mathfrak{L} = 0$.

3-24. 令 $C \subset [0,1] \times [0,1]$ 是所有 $\{p/q\} \times [0,1/q]$ 之并，其中 p/q 是 $[0,1]$ 中化为既约分数的有理数. 利用 C 证明习题 3-23 中的“测度”不能改成“容度”.

3-25. 对 n 用归纳法证明，若对一切 i, $a_i < b_i$，则 $[a_1,b_1] \times \cdots \times [a_n,b_n]$ 不是测度 0（或容度 0）的集.

3-26. 令 $f:[a,b] \to \mathbf{R}$ 为可积且非负，再令 $A_f = \{(x,y): a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$. 证明 A_f 为约当可测且有面积 $\int_a^b f$.

3-27. 若 $f:[a,b] \times [a,b] \to \mathbf{R}$ 为连续，证明

$$\int_a^b \int_a^y f(x,y)\,dx\,dy = \int_a^b \int_x^b f(x,y)\,dy\,dx.$$

提示：对一个适当的集 $C \subset [a,b] \times [a,b]$，用两种不同方法求 $\int_C f$.

3-28. *设 $D_{1,2}f$ 与 $D_{2,1}f$ 都连续，应用富比尼定理对 $D_{1,2}f = D_{2,1}f$ 给一简证. 提示：若 $D_{1,2}f(a) - D_{2,1}f(a) > 0$，必存在一个包含 a 的矩形 A，在其上 $D_{1,2}f - D_{2,1}f > 0$.

3-29. 设 $\mathbf{R}^3$ 的一个集是由 yz 平面上一个约当可测集绕 z 轴旋转而成，用富比尼定理导出其体积公式.

3-30. 令 C 为习题 1-17 中的集，证明

$$\int_{[0,1]}\left(\int_{[0,1]} \chi_C(x,y)\,dx\right)dy = \int_{[0,1]}\left(\int_{[0,1]} \chi_C(x,y)\,dy\right)dx = 0,$$

但 $\int_{[0,1]\times[0,1]} \chi_C$ 不存在.

3-31. 若 $A = [a_1,b_1] \times \cdots \times [a_n,b_n]$ 而 $f: A \to \mathbf{R}$ 为连续，定义 $F: A \to \mathbf{R}$ 为

$$F(x) = \int_{[a_1,x^1]\times\cdots\times[a_n,x^n]} f.$$

当 x 为 A 之内点时，$D_i F(x)$ 取何值?

3-32. *令 $f:[a,b] \times [c,d] \to \mathbf{R}$ 连续，并设 $D_2 f$ 连续. 定义 $F(y) = \int_a^b f(x,y)\,dx$. 证明莱布尼兹(Leibnitz) 规则 $F'(y) = \int_a^b D_2 f(x,y)\,dx$. 提示：$F(y) = \int_a^b f(x,y)\,dx = \int_a^b \left(\int_c^y D_2 f(x,\eta)\,d\eta + f(x,c)\right)dx$

(其证明将表明$D_2 f$的连续性可用弱得多的假设所代替.)

3-33. 若f: $[a, b] \times [c, d] \to \mathbf{R}$且$D_2 f$连续. 令$F(x,y) = \int_a^x f(t,y)\,dt$.

(a) 求$D_1 F$和$D_2 F$.

(b) 若$G(x) = \int_a^{g(x)} f(t,x)\,dt$,求$G'(x)$.

3-34. *令$g_1, g_2: \mathbf{R}^2 \to \mathbf{R}$为连续可微并设$D_1 g_2 = D_2 g_1$. 如习题2-21那样，令

$$f(x,y) = \int_0^x g_1(t,0)\,dt + \int_0^y g_2(x,t)\,dt.$$

证明$D_1 f(x,y) = g_1(x,y)$.

3-35. *(a) 令g为$\mathbf{R}^n \to \mathbf{R}^n$以下几种类型之一的线性变换:

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases} \quad \begin{cases} g(e_j) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases} \quad \begin{cases} g(e_k) = e_k & k \neq i,j \\ g(e_i) = e_j \\ g(e_j) = e_i. \end{cases}$$

若U是一个矩形，证明$g(U)$的体积是$|\det g| \cdot v(U)$.

(b) 证明对于任意线性变换$g: \mathbf{R}^n \to \mathbf{R}^n$，$g(U)$之体积是$|\det g| \cdot v(U)$.

提示：若$\det g \neq 0$，则g是(a)中考虑的那些类型的线性变换的复合.

3-36. *(卡伐列里(Cavalieri)原理). 令A和B是$\mathbf{R}^3$的约当可测子集. 令$A_c = \{(x,y):(x,y,c) \in A\}$，类似地定义$B_c$. 设每个$A_c$与$B_c$均为约当可测并有相同面积. 证明$A$与$B$体积相同.

3.5 单位分解

我们将在这一节里介绍积分理论中一个极其重要的工具.

定理3-11 令$A \subset \mathbf{R}^n$而$\mathcal{O}$是A的一个开覆盖. 于是必有C^∞函数φ的一个集合Φ，φ定义在一个包含A的开集上，且有以下性质:

(1) 对于每个$x \in A$我们有$0 \leq \varphi(x) \leq 1$.

(2) 对于每个$x \in A$均有一个含x的开集V使在其上只有有限多个$\varphi \in \Phi$不为0.

(3) 对于每个 $x \in A$，我们有 $\sum_{\varphi \in \Phi} \varphi(x) = 1$ (由(2)，对每个 x，在某个含 x 的开集中这个和是有限和).

(4) 对于每个 $\varphi \in \Phi$，均有 $\mathcal{O}$ 中一个开集 U 使 φ 在 U 内的某个闭子集外 $\varphi = 0$.

(一组满足(1)到(3)的函数集合 Φ 称为 A 的 C^∞ **单位分解**. 若 Φ 也满足(4)，就说它**从属**于 $\mathcal{O}$. 在本章中，我们只用到函数 φ 的连续性.)

证　情况1. A 为紧的.

这时 $\mathcal{O}$ 中有限个开集 $U_1, \cdots, U_n$ 覆盖 A. 显然只要作出一个从属于覆盖 $\{U_1, \cdots, U_n\}$ 的单位分解就够了. 我们先构造紧集 $D_i \subset U_i$ 使其内域覆盖 A. 集 D_i 可以用归纳法用如下方法构造. 设 $D_1, \cdots, D_k$ 已经构造好使得 $\{D_1$ 的内域$, \cdots, D_k$ 的内域$, U_{k+1}, \cdots, U_n\}$ 覆盖 A. 令

$$C_{k+1} = A - (\text{int } D_1 \cup \cdots \cup \text{int } D_k \cup U_{k+2} \cup \cdots \cup U_n).$$

于是 $C_{k+1} \subset U_{k+1}$ 而且是紧的. 因此可以找到一个紧集 D_{k+1} (习题1-22) 使得

$$C_{k+1} \subset (D_{k+1} \text{ 的内域}) \text{ 而且 } D_{k+1} \subset U_{k+1}.$$

作出 $D_1, \cdots, D_n$ 以后，令 ψ_i 是一个非负 C^∞ 函数，在 D_i 上为正，在某个含在 U_i 内的闭集之外为0(习题2-26). 因为 $\{D_1, \cdots, D_n\}$ 覆盖 A，故对某个包含 A 的开集 U 中一切 x 点，有 $\psi_1(x) + \cdots + \psi_n(x) > 0$. 在 U 上定义

$$\varphi_i(x) = \frac{\psi_i(x)}{\psi_1(x) + \cdots + \psi_n(x)}.$$

若 $f: U \to [0,1]$ 是一个 C^∞ 函数，在 A 上为1而在含在 U 中的某个闭集之外为0，则 $\Phi = \{f \cdot \varphi_1, \cdots, f \cdot \varphi_n\}$ 即为所求的单位分解.

情况2. $A = A_1 \cup A_2 \cup A_3 \cup \cdots$，其中每个 A_i 为紧而且 $A_i \subset (A_{i+1}$ 的内域).

对每个 i，令 $\mathcal{O}_i$ 由 $\mathcal{O}$ 中一切 U 所作的 $U \cap (\text{int } A_{i+1} - A_{i-2})$ 组成. 于是 $\mathcal{O}_i$ 是紧集 $B_i = A_i - \text{int } A_{i-1}$ 的开覆盖. 由情况1有 B_i 的从属于 $\mathcal{O}_i$ 的单位分解 Φ_i. 对于 $x \in A$，和数

$$\sigma(x) = \sum_{\varphi \in \Phi_i, \text{一切} i} \varphi(x)$$

在某个包含 x 的开集上为有限和，这是由于，若 $x \in A_i$，对 $\varphi \in \Phi_j$ 且 $j \geqslant i+2$，我们有 $\varphi(x) = 0$．对于每一 Φ_i 中的每个 φ，定义 $\varphi'(x) = \varphi(x)/\sigma(x)$．一切 φ'之集即为所求的单位分解.

情况 3. A 为开集.

令 $A_i = \{x \in A: |x| \leqslant i$ 且 x 到 A 边界的距离 $\geqslant 1/i\}$，再应用情况 2.

情况 4. A 为任意.

令 B 为 $\mathcal{O}$ 中一切 U 之并集. 由情况 3，必有 B 的单位分解，它也是 A 的单位分解. ▌

定理的条件(2)的一个重要推论应该引起注意. 令 $C \subset A$ 是紧的. 对每一个 $x \in C$ 均有一个包含 x 的开集 V_x，使得只有有限多个 $\varphi \in \Phi$ 在 V_x 上不为 0. 因为 C 是紧的，有限多个这样的 V_x 即可覆盖 C. 于是只有有限多个 $\varphi \in \Phi$ 在 C 上不为 0.

单位分解的一个重要应用可以说明它的主要作用 —— 把局部得到的结果拼接起来. 开集 $A \subset \mathbf{R}^n$ 的一个开覆盖 $\mathcal{O}$ 称为**可容许的**，如果每一个 $U \in \mathcal{O}$ 都包含在 A 内. 若 $\Phi = \{\varphi\}$ 为从属于 $\mathcal{O}$ 的单位分解，$f: A \to \mathbf{R}$ 在 A 之每点的某开集内都有界而且 $\{x: f$ 在 x 不连续$\}$ 有测度 0，则此时每一个 $\int_A \varphi \cdot |f|$ 都存在. 如果 $\sum_{\varphi \in \Phi} \int_A \varphi \cdot |f|$ 收敛（定理 3-11 的证明说明这些 φ 可以排成一个序列），我们就定义 f（广义）**可积**. 这隐含着 $\sum_{\varphi \in \Phi} \left| \int_A \varphi \cdot f \right|$ 收敛. 亦即 $\sum_{\varphi \in \Phi} \int_A \varphi \cdot f$ 绝对收敛. 我们把它定义为 $\int_A f$. 这些定义都与 $\mathcal{O}$ 和 Φ 无关（但见习题 3-28）.

定理 3-12

(1) 若 $\Psi = \{\psi\}$ 是另一个从属于 A 的可容许开覆盖 $\mathcal{O}'$ 的单位分解，而且 $\{x: f$ 在 x 不连续$\}$ 为测度 0 的集，则 $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$ 也收敛，且

$$\sum_{\psi \in \Phi} \int_A \varphi \cdot f = \sum_{\psi \in \Psi} \int_A \psi \cdot f.$$

（2）若 A 和 f 有界，则 f 广义可积.

（3）若 A 为约当可测而 f 为有界可积，则 $\int_A f$ 的这个定义与原来的一致.

证

（1）因为除了在某紧集 C 上之外，$\varphi \cdot f = 0$，而且只有有限多个 ψ 在 C 上非零，我们可以写出

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\varphi \in \Phi} \int_A \sum_{\psi \in \Psi} \psi \cdot \varphi \cdot f = \sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \int_A \psi \cdot \varphi \cdot f.$$

将此结果用于 $|f|$，即得 $\sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \int_A \psi \cdot \varphi \cdot |f|$ 的收敛性，从而又有 $\sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \left| \int_A \psi \cdot \varphi \cdot f \right|$ 的收敛性. 这个绝对收敛性确保了在上面等式中可交换求和次序，很明显所得的二重求和式等于 $\sum_{\psi \in \Psi} \int_A \psi \cdot f$. 最后，将此结果应用于 $|f|$ 就证明了 $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$ 的收敛性.

（2）若 A 含于闭矩形 B 中，而且对 $x \in A$ 有 $|f(x)| \leqslant M, F \subset \Phi$ 又是有限的，这时

$$\sum_{\varphi \in F} \int_A \varphi \cdot |f| \leqslant \sum_{\varphi \in F} M \int_A \varphi = M \int_A \sum_{\varphi \in F} \varphi \leqslant Mv(B),$$

因此在 A 上 $\sum_{\varphi \in F} \varphi \leqslant 1$.

（3）若 $\varepsilon > 0$，必有（习题3-22）一个紧的约当可测集 $C \subset A$ 使 $\int_{A-C} 1 < \varepsilon$. 只有有限多个 $\varphi \in \Phi$ 在 C 上不为0. 若 $F \subset \Phi$ 是包含这些 φ 的任意有限集合，而 $\int_A f$ 具有原来的意义，则

$$\left| \int_A f - \sum_{\varphi \in F} \int_A \varphi \cdot f \right| \leqslant \int_A \left| f - \sum_{\varphi \in F} \varphi \cdot f \right|$$

$$\leqslant M \int_A (1 - \sum_{\varphi \in F} \varphi) = M \int_A \sum_{\varphi \in \Phi - F} \varphi$$

$$\leqslant M\int_{A-C} 1 \leqslant M\varepsilon. \qquad \blacksquare$$

习题

3-37. (a) 设$f:(0,1)\to\mathbf{R}$是非负连续函数. 证明$\int_{(0,1)} f$当且仅当$\lim\limits_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} f$存在时存在.

(b) 令$A_n = [1-1/2^n, 1-1/2^{n+1}]$. 设$f:(0,1)\to\mathbf{R}$满足$\int_{A_n} f = (-1)^n/n$, 而当$x \notin$ (任意A_n) 时$f(x)=0$. 证明$\int_{(0,1)} f$不存在, 但$\lim\limits_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} f = \ln 2$.

3-38. 令A_n为含于$(n, n+1)$中的一个闭集. 设$f:\mathbf{R}\to\mathbf{R}$满足$\int_{A_n} f = (-1)^n/n$而当$x \notin$ (任意A_n) 时$f=0$. 求两个单位分解Φ和Ψ使得$\sum\limits_{\varphi\in\Phi}\int_{\mathbf{R}}\varphi\cdot f$与$\sum\limits_{\psi\in\Psi}\int_{\mathbf{R}}\psi\cdot f$分别绝对收敛于不同值.

3.6 变量替换

若$g:[a,b]\to\mathbf{R}$连续可微而$f:\mathbf{R}\to\mathbf{R}$连续, 那么, 众所周知

$$\int_{g(a)}^{g(b)} f = \int_a^b (f\circ g)\cdot g'.$$

证明很简单: 若$F'=f$, 则$(F\circ g)' = (f\circ g)\cdot g'$, 于是上式左方是$F(g(b)) - F(g(a))$, 而右方则是$F\circ g(b) - F\circ g(a) = F(g(b)) - F(g(a))$.

我们留请读者证明, 若g是1-1的, 则上面的公式可以写为

$$\int_{g((a,b))} f = \int_{(a,b)} f\circ g\cdot |g'|.$$

(分别考虑g为递增与g为递减的两种情况.) 这个公式对高维的推广则不是显而易见的.

定理 3-13　令 $A \subset \mathbf{R}^n$ 为一开集，$g: A \to \mathbf{R}^n$ 是1-1的连续可微函数，使得对于一切 $x \in A$，$\det g'(x) \neq 0$. 若 $f: g(A) \to \mathbf{R}$ 是一个可积函数，则

$$\int_{g(A)} f = \int_A (f \circ g) \,|\det g'|.$$

证　我们从一些重要的简化情形开始.

1　设 A 有一个可容许开覆盖 $\mathcal{O}$，使对每个 $U \in \mathcal{O}$ 以及任意的可积函数 f，我们都有

$$\int_{g(U)} f = \int_U (f \circ g) \,|\det g'|.$$

这时定理必在整个 A 上成立（因为 g 在围绕每点的某开集上自动地是1-1的，定理中只在这一部分用到 g 在整个 A 上是1-1的，这是不足为奇的）.

(1)的证明. 所有 $g(U)$ 的集是 $g(A)$ 的一个开覆盖. 令 Φ 为从属于这个覆盖的单位分解. 若在 $g(U)$ 之外 $\varphi = 0$，则由于 g 是1-1的，$(\varphi \cdot f) \circ g$ 在 U 之外为0. 所以，由等式

$$\int_{g(U)} \varphi \cdot f = \int_U [(\varphi \cdot f) \circ g] \,|\det g'|$$

可以得到

$$\int_{g(A)} \varphi \cdot f = \int_A [(\varphi \cdot f) \circ g] \,|\det g'|.$$

因此

$$\int_{g(A)} f = \sum_{\varphi \in \Phi} \int_{g(A)} \varphi \cdot f = \sum_{\varphi \in \Phi} \int_A [(\varphi \cdot f) \circ g] \,|\det g'|$$

$$= \sum_{\varphi \in \Phi} \int_A (\varphi \circ g)(f \circ g) \,|\det g'| = \int_A (f \circ g) \,|\det g'|.$$

注. 若设对 $g(A)$ 的某个可容许覆盖中的 V，有

$$\int_V f = \int_{g^{-1}(V)} (f \circ g) \,|\det g'|,$$

也可得此定理. 这只要把(1)用到 g^{-1} 上去就可以了.

2 本定理仅须对$f=1$加以证明

(2) 的证明 若定理对于$f=1$成立，则它对于常值函数也成立. 令V为$g(A)$中的一个矩形，P是V的一个分法. 对P的每个子矩形S，令f_S为常值函数$m_S(f)$. 这时

$$\begin{aligned} L(f,P) &= \sum_S m_S(f)\cdot v(S) = \sum_S \int_{\text{int } S} f_S \\ &= \sum_S \int_{g^{-1}(\text{int } S)} (f_S \circ g)\,|\det g'| \\ &\leqslant \sum_S \int_{g^{-1}(\text{int } S)} (f\circ g)\,|\det g'| \\ &= \int_{g^{-1}(V)} (f\circ g)\,|\det g'| \end{aligned}$$

因为$\int_V f$是一切函数$L(f,P)$的上确界，由此得证$\int_V f \leqslant \int_{g^{-1}(V)} (f\circ g)\,|\det g'|$. 令$f_S = m_S(f)$并作类似论证，又有$\int_V f \geqslant \int_{g^{-1}(V)} (f\circ g)\,|\det g'|$. 用上面的注即得所要结果.

3 如果定理对$g:A\to\mathbf{R}^n$和$h:B\to\mathbf{R}^n$都成立，而且$g(A)\subset B$，则它对$h\circ g:A\to\mathbf{R}^n$也成立.

(3)的证明.

$$\begin{aligned} \int_{h\circ g(A)} f &= \int_{h(g(A))} f = \int_{g(A)} (f\circ h)\,|\det h'| \\ &= \int_A [(f\circ h)\circ g]\cdot[\,|\det h'|\circ g]\,|\det g'| \\ &= \int_A f\circ(h\circ g)\,|\det(h\circ g)'|. \end{aligned}$$

4 若g是线性变换则定理成立.

(4)的证明. 由(1)和(2)只须对任意开矩形U证明

$$\int_{g(U)} 1 = \int_U |\det g'|$$

即可. 这就是习题3-35.

把(3)和(4)放在一起考察，可以假设，对任一特定的$a\in A$，$g'(a)$是单位矩阵. 事实上，若T是线性变换$Dg(a)$，则

$(T^{-1}\circ g)'(a)=I$. 因为这定理对 T 成立，所以若它对 $T^{-1}\circ g$ 成立则对 g 也成立.

我们现已作好了给出证明的准备，在此对 n 进行归纳. 定理陈述前的说明连同(1)，(2)两点已经证明了 $n=1$ 的情况. 设定理在维数为 $n-1$ 时成立，我们要证明定理在 n 维时也成立. 对于每一点 $a\in A$，我们只须找到一个开集 U，使 $a\in U\subset A$，而且定理对 U 成立即可. 此外，我们还可以假设 $g'(a)=I$.

定义 $h:A\to\mathbf{R}^n$ 为 $h(x)=(g^1(x),\cdots,g^{n-1}(x),x^n)$. 于是 $h'(a)=I$. 所以在某个开集 U'中函数 h 是1-1 的而且 $\det h'(x)\neq 0$，这里 $a\in U'\subset A$. 于是我们可以定义 $k:h(U')\to\mathbf{R}^n$ 为 $k(x)=(x^1,\cdots,x^{n-1},g^n(h^{-1}(x)))$ 而 $g=k\circ h$. 于是我们把 g 表示成了两个映射的复合，而其中每一个都只改变少于 n 个坐标(图 3-3).

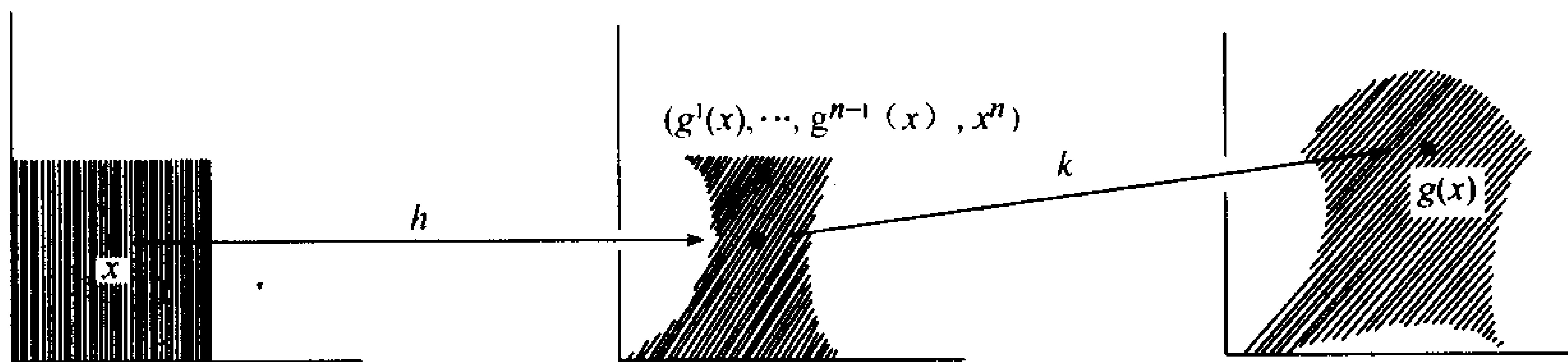


图 3-3

必须注意到几个细节，以确保 k 正是那一类适当的函数. 因为

$$(g^n\circ h^{-1})'(h(a))=(g^n)'(a)\cdot[h'(a)]^{-1}=(g^n)'(a),$$

所以 $D_n(g^n\circ h^{-1})(h(a))=D_ng^n(a)=1$，因此 $k'(h(a))=I$. 于是在某个开集 V 上 $(h(a)\in V\subset h(U'))$，函数 k 是1-1 的而且 $\det k'(x)\neq 0$. 令 $U=k^{-1}(V)$ 就有 $g=k\circ h$，这里 $h:U\to\mathbf{R}^n$，$k:V\to\mathbf{R}^n$，且 $h(U)\subset V$. 由(3)，只须对 h 和 k 证明定理就行了. 我们给出对 h 的证明，对 k 的证明类似，而且更容易些.

令 $W\subset U$ 是形如 $D\times[a_n,b_n]$ 的矩形，其中 D 是 $\mathbf{R}^{n-1}$ 中的矩形. 由富比尼定理

$$\int_{h(W)}1=\int_{[a_n,b_n]}\left(\int_{h(D\times\{x^n\})}1\,\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\right)\mathrm{d}x^n.$$

令 $h_{x^n}:D\to\mathbf{R}^{n-1}$ 定义为 $h_{x^n}(x^1,\cdots,x^{n-1}) = (g^1(x^1,\cdots,x^n),\cdots,g^{n-1}(x^1,\cdots,x^n))$. 则每一个 h_{x^n}显然都是 1-1 的，而且

$$\det(h_{x^n})'(x^1,\cdots,x^{n-1}) = \det h'(x^1,\cdots,x^n) \neq 0.$$

此外

$$\int_{h(D\times\{x^n\})} 1\,\mathrm{d}x^1\cdots\mathrm{d}x^{n-1} = \int_{h_{x^n}(D)} 1\,\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}.$$

应用 $n-1$ 维情况下的定理，于是就得到

$$\begin{aligned}\int_{h(W)} 1 &= \int_{[a_n,b_n]}\left(\int_{h_{x^n}(D)} 1\,\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\right)\mathrm{d}x^n\\ &= \int_{[a_n,b_n]}\left(\int_D |\det(h_{x^n})'(x^1,\cdots,x^{n-1})|\,\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\right)\mathrm{d}x^n\\ &= \int_{[a_n,b_n]}\left(\int_D |\det h'(x^1,\cdots,x^n)|\,\mathrm{d}x^1\cdots\mathrm{d}x^{n-1}\right)\mathrm{d}x^n\\ &= \int_W |\det h'|. \qquad \blacksquare\end{aligned}$$

用下面的定理就可以在定理 3-13 的假设中除掉 $\det g'(x)\neq 0$ 的条件. 下面的定理常起意想不到的作用.

定理 3-14（萨德（Sard）定理） 令 $g:A\to\mathbf{R}^n$ 为连续可微，其中 $A\subset\mathbf{R}^n$ 是一个开集，再令 $B=\{x\in A:\det g'(x)=0\}$. 则 $g(B)$ 具有测度 0.

证 令 $U\subset A$ 是一个闭矩形，譬如说，其各边之长均为 l. 令 $\varepsilon>0$. 若 N 充分大，将 U 分成 N^n 个边长为 l/N 的矩形，则对其中任一个矩形 S，若 $x\in S$ 时，对于一切 $y\in S$，我们有

$$|Dg(x)(y-x)-(g(y)-g(x))| < \varepsilon|x-y| \leqslant \varepsilon\sqrt{n}(l/N).$$

若 S 与 B 相交，可取 $x\in S\cap B$. 因为 $\det g'(x)=0$，而集 $\{Dg(x)(y-x):y\in S\}$ 又位于 $\mathbf{R}^n$ 的一个 $n-1$ 维子空间 V 中. 所以 $\{g(y)-g(x):y\in S\}$ 与 V 距离小于 $\varepsilon\sqrt{n}(l/N)$，而 $\{g(y):y\in S\}$ 与 $n-1$ 维平面 $V+g(x)$ 距离小于 $\varepsilon\sqrt{n}(l/N)$. 另一方面，由引理 2-10，必有常数 M 使得

$$|g(x)-g(y)|<M|x-y|\leqslant M\sqrt{n}(l/N).$$

所以，若 S 与 B 相交，则集 $\{g(y):y\in S\}$ 包含在一个柱体之内，其高 $<2\varepsilon\sqrt{n}(l/N)$，其底是半径小于 $M\sqrt{n}(l/N)$ 的 $n-1$ 维球. 这个柱体的体积小于 $C(l/N)^n\varepsilon$，C 是某个常数. 这种矩形 S 最多有 N^n 个，所以 $g(U\cap B)$ 包含在体积小于 $C(l/N)^n\cdot\varepsilon\cdot N^n=Cl^n\cdot\varepsilon$ 的一个集内. 因为此式对任意 $\varepsilon>0$ 都对，所以 $g(U\cap B)$ 有测度 0. 又因（习题 3-13）我们可用可数多个这样的矩形 U 覆盖 A，由定理 3-14 即得所要的结果. ▌

定理 3-14 其实只是萨德定理较容易的一部分. 更深的结果的叙述和证明可以在参考文献［17］的第 47 页上找到.

习题

3-39. 用定理 3-14 证明定理 3-13，去掉假设 $\det g'(x)\neq 0$.

3-40. 若 $g:\mathbf{R}^n\to\mathbf{R}^n$ 而且 $\det g'(x)\neq 0$，证明在某个包含 x 的开集中可以写成 $g=T\circ g_n\circ\cdots\circ g_1$，这里 g_i 有 $g_i(x)=(x^1,\cdots,f_i(x),\cdots,x^n)$ 之形，而 T 是一个线性变换. 证明，当且仅当 $g'(x)$ 为对角矩阵时，才可以写成 $g=g_n\circ\cdots\circ g_1$ 的形式.

3-41. 定义 $f:\{r:r>0\}\times(0,2\pi)\to\mathbf{R}^2$ 为 $f(r,\theta)=(r\cos\theta,r\sin\theta)$.

(a) 证明 f 是 1-1 的，计算 $f'(r,\theta)$ 并证明对一切 (r,θ) 均有 $\det f'(r,\theta)\neq 0$. 证明 $f(\{r:r>0\}\times(0,2\pi))$ 即是习题 2-23 中的集 A.

(b) 若 $P=f^{-1}$，证明 $P(x,y)=(r(x,y),\theta(x,y))$，其中

$$r(x,y)=\sqrt{x^2+y^2},$$

$$\theta(x,y)=\begin{cases}\arctan y/x & x>0,\ y>0,\\ \pi+\arctan y/x & x<0,\\ 2\pi+\arctan y/x & x>0,\ y<0,\\ \pi/2 & x=0,\ y>0,\\ 3\pi/2 & x=0,\ y<0.\end{cases}$$

（这里 arc tan 记作函数 $\tan:(-\pi/2,\pi/2)\to\mathbf{R}$ 的反函数.）求 $P'(x,y)$. 函数 P 称 A 上的**极坐标系**.

(c) 令 $C \subset A$ 是半径 $r_1, r_2(r_2 > r_1)$ 的圆弧和过 0 点而与 x 轴成 $\theta_1, \theta_2(\theta_2 > \theta_1)$ 角的半射线所围成的区域. 若 $h: C \to \mathbf{R}$ 而且 $h(x,y) = g(r(x,y), \theta(x,y))$，求证

$$\int_C h = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} rg(r,\theta)\,\mathrm{d}\theta\,\mathrm{d}r.$$

若 $B_r = \{(x,y): x^2 + y^2 \leqslant r^2\}$，证明

$$\int_{B_r} h = \int_0^r\int_0^{2\pi} rg(r,\theta)\,\mathrm{d}\theta\,\mathrm{d}r.$$

(d) 若 $C_r = [-r,r] \times [-r,r]$，证明

$$\int_{B_r} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \pi(1 - e^{-r^2})$$

而

$$\int_{C_r} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \left(\int_{-r}^{r} e^{-r^2}\mathrm{d}r\right)^2.$$

(e) 证明 $\lim\limits_{r\to\infty}\int_{B_r} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \lim\limits_{r\to\infty}\int_{C_r} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$ 而最终得到

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$

“数学家就是这样一种人，这个公式对他们就像二二得四对你一样明显. 刘维尔（Liouville）是一个数学家.”

——开尔文勋爵

第4章　链上的积分

4.1　代数预备知识

若 V 是($\mathbf{R}$ 上的)向量空间，我们用 V^k 来记 k 重乘积 $V\times\cdots\times V$. 函数 $T:V^k\to\mathbf{R}$, 如果对每个 i，$1\leqslant i\leqslant k$ 都有

$$T(v_1,\cdots,v_i+v_i',\cdots,v_k)=T(v_1,\cdots,v_i,\cdots,v_k)+T(v_1,\cdots,v_i',\cdots,v_k),$$
$$T(v_1,\cdots,av_i,\cdots,v_k)=aT(v_1,\cdots,v_i,\cdots,v_k),$$

就称为**重线性函数**. **重线性**函数 $T:V^k\to\mathbf{R}$ 称为 V 上的 k **阶张量**，而全体 k 阶张量之集，记作 $\mathfrak{T}^k(V)$，成为一个($\mathbf{R}$ 上的)向量空间，其中规定，对于 $S,T\in\mathfrak{T}^k(V)$, $a\in\mathbf{R}$ 有

$$(S+T)(v_1,\cdots,v_k)=S(v_1,\cdots,v_k)+T(v_1,\cdots,v_k),$$
$$(aS)(v_1,\cdots,v_k)=a\cdot S(v_1,\cdots,v_k).$$

还有一个把各个空间 $\mathfrak{T}^k(V)$ 联结起来的运算. 若 $S\in\mathfrak{T}^k(V)$, $T\in\mathfrak{T}^l(V)$, 我们定义其**张量积** $S\otimes T\in\mathfrak{T}^{k+l}(V)$ 为

$$S\otimes T(v_1,\cdots,v_k,v_{k+1},\cdots,v_{k+l})=S(v_1,\cdots,v_k)\cdot T(v_{k+1},\cdots,v_{k+l}).$$

注意，在这里因子 S 和 T 的次序是紧要的，因为 $S\otimes T$ 和 $T\otimes S$ 远非一回事. $\otimes$的下列性质作为容易的练习留给读者.

$$(S_1+S_2)\otimes T=S_1\otimes T+S_2\otimes T,$$
$$S\otimes(T_1+T_2)=S\otimes T_1+S\otimes T_2,$$
$$(aS)\otimes T=S\otimes(aT)=a(S\otimes T),$$
$$(S\otimes T)\otimes U=S\otimes(T\otimes U).$$

$(S\otimes T)\otimes U$ 和 $S\otimes(T\otimes U)$ 通常简记作 $S\otimes T\otimes U$，更高阶的乘积 $T_1\otimes\cdots\otimes T_r$ 定义类似.

读者可能已经注意到，$\mathfrak{T}^1(V)$ 正是对偶空间 V^*. 用运算 $\otimes$ 可以把其他的向量空间 $\mathfrak{T}^k(V)$ 用 $\mathfrak{T}^1(V)$ 表示出来.

定理 4-1　令 $v_1,\cdots,v_n$ 是 V 的一个基底，$\varphi_1,\cdots,\varphi_n$ 为其对偶基底：$\varphi_i(v_j)=\delta_{ij}$. 于是所有的 k 重张量积

$$\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\qquad 1\leqslant i_1,\cdots,i_k\leqslant n$$

之集是 $\mathfrak{T}^k(V)$ 的一个基底，故 $\mathfrak{T}^k(V)$ 的维数是 n^k.

证　注意

$$\begin{aligned}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}(v_{j_1},\cdots,v_{j_k}) &= \delta_{i_1j_1}\cdot\cdots\cdot\delta_{i_kj_k}\\ &=\begin{cases}1 & \text{若 } j_1=i_1,\cdots,j_k=i_k,\\ 0 & \text{其他情况}.\end{cases}\end{aligned}$$

若 $w_1,\cdots,w_k$ 是 k 个向量 $w_i=\sum_{j=1}^{n}a_{ij}v_j$，而 T 在 $\mathfrak{T}^k(V)$ 中，则

$$\begin{aligned}T(w_1,\cdots,w_k) &= \sum_{j_1,\cdots,j_k=1}^{n}a_{1,j_1}\cdots a_{k,j_k}T(v_{j_1},\cdots,v_{j_k})\\ &=\sum_{i_1,\cdots,i_k=1}^{n}T(v_{i_1},\cdots,v_{i_k})\cdot\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}(w_1,\cdots,w_k).\end{aligned}$$

于是 $T=\sum_{i_1,\cdots,i_k=1}^{n}T(v_{i_1},\cdots,v_{i_k})\cdot\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}$. 从而，$\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}$ 张成 $\mathfrak{T}^k(V)$.

现在设有实数 $a_{i_1,\cdots,i_k}$ 存在，使得

$$\sum_{i_1,\cdots,i_k=1}^{n}a_{i_1,\cdots,i_k}\cdot\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}=0.$$

将此式左右两边都作用到 $(v_{j_1},\cdots,v_{j_k})$ 上去得 $a_{j_1,\cdots,j_k}=0$. 于是 $\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}$ 是线性无关的. ▌

我们在对偶空间情况下已经熟悉的一个重要的作法，也可以对张量来作. 若 $f:V\to W$ 是一个线性变换，我们定义线性变换 f^*:

$\mathfrak{T}^k(W) \to \mathfrak{T}^k(V)$ 为

$$f^* T(v_1, \cdots, v_k) = T(f(v_1), \cdots, f(v_k)).$$

这里 $T \in \mathfrak{T}^k(W), v_1, \cdots, v_k \in V$. 容易验证 $f^*(S \otimes T) = f^* S \otimes f^* T$.

在 V^* 的元素之外，读者已经见识过一些张量．第一个例子是内积 $\langle , \rangle \in \mathfrak{T}^2(\mathbf{R}^n)$. 因为数学上每一个好东西都值得推广，我们定义 V 上的**内积**为一个**对称**、**正定**的 2 阶张量 T. 所谓对称是指 $T(v,w) = T(w,v), v, w \in V$, 所谓正定是指当 $v \neq 0$ 时，$T(v,v) > 0$. 我们特以 $\langle , \rangle$ 专指 $\mathbf{R}^n$ 上**通常的内积**．下面的定理说明，我们的推广并不失之过分空泛．

定理 4-2 若 T 是 V 上的一个内积，则必有 V 的一个基底 $v_1, \cdots, v_n$ 使 $T(v_i, v_j) = \delta_{ij}$.（这个基底称为 T 的**标准正交基底**.）因此必有一个同构 $f: \mathbf{R}^n \to V$ 使对 $x, y \in \mathbf{R}^n$ 有 $T(f(x), f(y)) = \langle x, y \rangle$. 换言之, $f^* T = \langle , \rangle$.

证 令 $w_1, \cdots, w_n$ 是 V 的任一基底．定义

$$w_1' = w_1,$$

$$w_2' = w_2 - \frac{T(w_1', w_2)}{T(w_1', w_1')} \cdot w_1',$$

$$w_3' = w_3 - \frac{T(w_1', w_3)}{T(w_1', w_1')} \cdot w_1' - \frac{T(w_2', w_3)}{T(w_2', w_2')} \cdot w_2',$$

等等．很容易验证当 $i \neq j$ 时 $T(w_i', w_j') = 0$, 又 $w_i' \neq 0$, 因此 $T(w_i', w_i') > 0$. 现在定义 $v_i = w_i' / \sqrt{T(w_i', w_i')}$．至于同构 f, 则可由 $f(e_i) = v_i$ 定义．▌

内积尽管重要，其作用却远不如另一个人们熟悉的、几乎无处不在的函数，即张量 $\det \in \mathfrak{T}^n(\mathbf{R}^n)$[1] 在打算推广这个函数时，我们回想一下，交换矩阵的两行会改变其行列式的符号．这一点启发我们作下面的定义．若对 k 阶张量 $\omega \in \mathfrak{T}^k(V)$, 有

1. $\det \in \mathfrak{T}(\mathbf{R}^n)$ 的定义是 $\det((a_{11}, \cdots, a_{1n}), \cdots, (a_{n1}, \cdots, a_{nn})) = \det(a_{ij})$. 以下会看到，它是一个 n 阶交代张量: $\det \in \Omega^n(\mathbf{R}^n)$. —译者注

$$\omega(v_1,\cdots,v_i,\cdots,v_j,\cdots,v_k)=-\omega(v_1,\cdots,v_j,\cdots,v_i,\cdots,v_k)$$

对一切 $v_1,\cdots,v_k\in V$,

就称 ω 为**交代的**.（上式中 v_i 与 v_j 对换了，其他的 v 未动）. 一切交代的 h 阶张量之集显然是 $\mathfrak{T}^k(V)$ 的一子空间 $\Omega^k(V)$. 因为要作出一个行列式都相当费事，所以，k 阶交代张量也不容易写出来，这是并不奇怪的. 但是有一个统一的方法把它们全都表示出来. 回忆一下一个排列 σ 的符号 sgn σ 定义如下：当 σ 为偶时为 +1，当 σ 为奇时为 −1. 若 $T\in\mathfrak{T}^k(V)$, 我们定义 Alt(T) 为

$$\mathrm{Alt}(T)(v_1,\cdots,v_k)=\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\,\sigma\cdot T(v_{\sigma(1)},\cdots,v_{\sigma(k)}),$$

S_k 是整数 1 到 k 的一切排列之集.

定理 4-3

(1) 若 $T\in\mathfrak{T}^k(V)$, 则 $\mathrm{Alt}(T)\in\Omega^k(V)$.

(2) 若 $\omega\in\Omega^k(V)$, 则 $\mathrm{Alt}(\omega)=\omega$.

(3) 若 $T\in\mathfrak{T}^k(V)$, 则 $\mathrm{Alt}(\mathrm{Alt}(T))=\mathrm{Alt}(T)$.

证

(1) 令 (i,j) 表示对换 i 和 j 而令其他数不动的排列. 若 $\sigma\in S_k$, 令 $\sigma'=\sigma\cdot(i,j)$. 于是

$$\begin{aligned}&\mathrm{Alt}(T)(v_1,\cdots,v_j,\cdots,v_i,\cdots,v_k)\\&=\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\,\sigma\cdot T(v_{\sigma(1)},\cdots,v_{\sigma(j)},\cdots,v_{\sigma(i)},\cdots,v_{\sigma(k)})\\&=\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\,\sigma\cdot T(v_{\sigma'(1)},\cdots,v_{\sigma'(i)},\cdots,v_{\sigma'(j)},\cdots,v_{\sigma'(k)})\\&=\frac{1}{k!}\sum_{\sigma'\in S_k}-\mathrm{sgn}\,\sigma'\cdot T(v_{\sigma'(1)},\cdots,v_{\sigma'(k)})\\&=-\mathrm{Alt}(T)(v_1,\cdots,v_k).\end{aligned}$$

(2) 若 $\omega\in\Omega^k(V)$, 且 $\sigma=(i,j)$, 则 $\omega(v_{\sigma(1)},\cdots,v_{\sigma(k)})=\mathrm{sgn}\,\sigma\cdot\omega(v_1,\cdots,v_k)$. 因为每一个 $\sigma\in S_k$ 都是许多个 (i,j) 的积，上式对一切 $\sigma\in S_k$ 都成立. 所以

$$\begin{aligned}\mathrm{Alt}(\omega)(v_1,\cdots,v_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\,\sigma\cdot\omega(v_{\sigma(1)},\cdots,v_{\sigma(k)})\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\,\sigma\cdot\mathrm{sgn}\,\sigma\cdot\omega(v_1,\cdots,v_k)\\ &= \omega(v_1,\cdots,v_k).\end{aligned}$$

(3) 由(1),(2)直接可得. ▌

为了确定 $\Omega^k(V)$ 的维数，我们希望有一个类似于定理 4-1 的定理. 若 $\omega\in\Omega^k(V)$，$\eta\in\Omega^l(V)$，$\omega\otimes\eta$ 通常当然不在 $\Omega^{k+l}(V)$ 中. 所以我们要定义一个新的乘积，即**楔积** $\omega\wedge\eta\in\Omega^{k+l}(V)$. 定义如下：

$$\omega\wedge\eta=\frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega\otimes\eta).$$

(为何要乘以式子前面这个奇怪的系数，以后我们将会看出来.) $\wedge$ 的下列性质留给读者作为练习：

$$\begin{gathered}(\omega_1+\omega_2)\wedge\eta=\omega_1\wedge\eta+\omega_2\wedge\eta,\\ \omega\wedge(\eta_1+\eta_2)=\omega\wedge\eta_1+\omega\wedge\eta_2,\\ a\omega\wedge\eta=\omega\wedge a\eta=a(\omega\wedge\eta),\\ \omega\wedge\eta=(-1)^{kl}\eta\wedge\omega,\\ f^*(\omega\wedge\eta)=f^*(\omega)\wedge f^*(\eta).\end{gathered}$$

等式 $(\omega\wedge\eta)\wedge\theta=\omega\wedge(\eta\wedge\theta)$ 也成立，但证起来比较麻烦.

定理 4-4

(1) 若 $S\in\mathfrak{T}^k(V)$，$T\in\mathfrak{T}^l(V)$ 而且 $\mathrm{Alt}(S)=0$，则

$$\mathrm{Alt}(S\otimes T)=\mathrm{Alt}(T\otimes S)=0.$$

(2)
$$\begin{aligned}\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta) &= \mathrm{Alt}(\omega\otimes\eta\otimes\theta)\\ &= \mathrm{Alt}(\omega\otimes\mathrm{Alt}(\eta\otimes\theta)).\end{aligned}$$

(3) 若 $\omega\in\Omega^k(V)$，$\eta\in\Omega^l(V)$，而 $\theta\in\Omega^m(V)$，则

$$\begin{aligned}(\omega\wedge\eta)\wedge\theta &= \omega\wedge(\eta\wedge\theta)\\ &= \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega\otimes\eta\otimes\theta).\end{aligned}$$

证

(1) $\mathrm{Alt}(S \otimes T)(v_1, \cdots, v_{k+l})$

$$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

$$\cdot T(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

若 $G \subset S_{k+l}$ 包含一切使 $k+1, \cdots, k+l$ 不动的排列 σ，则

$$\sum_{\sigma \in G} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$
$$= \Big[\sum_{\sigma' \in S_k} \operatorname{sgn} \sigma' \cdot S(v_{\sigma'(1)}, \cdots, v_{\sigma'(k)}) \Big] \cdot T(v_{k+1}, \cdots, v_{k+l}) = 0.$$

现在设 $\sigma_0 \notin G$. 令 $G \cdot \sigma_0 = \{\sigma \cdot \sigma_0 : \sigma \in G\}$ 并令 $v_{\sigma_0(1)}, \cdots, v_{\sigma_0(k+l)} = w_1, \cdots, w_{k+l}$. 于是

$$\sum_{\sigma \in G \cdot \sigma} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$
$$= \Big[\operatorname{sgn} \sigma_0 \cdot \sum_{\sigma' \in G} \operatorname{sgn} \sigma' \cdot S(w_{\sigma'(1)}, \cdots, w_{\sigma'(k)}) \Big]$$
$$\cdot T(w_{k+1}, \cdots, w_{k+l}) = 0.$$

注意 $G \cap G \cdot \sigma_0 = \varnothing$. 实际上，若有 $\sigma \in G \cap G \cdot \sigma_0$，则对某 $\sigma' \in G$，$\sigma = \sigma' \cdot \sigma_0$，又有 $\sigma_0 = (\sigma')^{-1} \cdot \sigma \in G$，这就是矛盾. 如此继续，可把 S_{k+l} 分为互不相交的子集. 在每个子集上和均为 0，所以对 S_{k+l} 的总和也是 0，$\mathrm{Alt}(T \otimes S) = 0$ 的证明类似.

(2) 我们有

$\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta) = \mathrm{Alt}(\eta \otimes \theta) - \mathrm{Alt}(\eta \otimes \theta) = 0.$

所以由 (1) 有

$$0 = \mathrm{Alt}(\omega \otimes [\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta])$$
$$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta)) - \mathrm{Alt}(\omega \otimes \eta \otimes \theta).$$

类似地可证明另一个等式.

(3) $(\omega \wedge \eta) \wedge \theta = \dfrac{(k+l+m)!}{(k+l)!\,m!} \mathrm{Alt}((\omega \wedge \eta) \otimes \theta)$

$$= \frac{(k+l+m)!}{(k+l)!m!}\frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta).$$

类似地可证明另一个等式．▌

$\omega \wedge (\eta \wedge \theta)$ 和 $(\omega \wedge \eta) \wedge \theta$ 自然地都简记作 $\omega \wedge \eta \wedge \theta$. 更高阶的积 $\omega_1 \wedge \cdots \wedge \omega_r$ 可类似地定义．若 $v_1, \cdots, v_n$ 是 V 的一个基底而 $\varphi_1, \cdots, \varphi_n$ 是其对偶基底，$\Omega^k(V)$ 的一个基底现在就很容易作出来了．

定理 4-5 $\Omega^k(V)$ 的一个基底是

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \quad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$$

全体的集，所以 $\Omega^k(V)$ 的维数是

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

证 若 $\omega \in \Omega^k(V) \subset \mathfrak{T}^k(V)$，则有

$$\omega = \sum_{i_1, \cdots, i_k} a_{i_1, \cdots, i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}.$$

于是

$$\omega = \mathrm{Alt}(\omega) = \sum_{i_1, \cdots, i_k} a_{i_1, \cdots, i_k} \mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}).$$

因为每一个 $\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$ 都是某一个 $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ 的常数倍，故张成 $\Omega^k(V)$. 线性无关的证明和定理 4-1 的一样（参看习题 4-1．）▌

若 V 之维数为 n，则由定理 4-5，推得 Ω^n (V) 的维数为 1. 所以 V 上一切 n 阶交代张量都是其中任一个非零 n 阶交代张量的倍数．由于行列式是 $\Omega^n(\mathbf{R}^n)$ 的元素之一例，下面的定理中会出现行列式也就不足为奇了．

定理 4-6 令 $v_1, \cdots, v_n$ 是 V 的一个基底，又令 $\omega \in \Omega^n(V)$. 若 $w_i = \sum_{j=1}^n a_{ij} v_j$ 是 V 中 n 个向量，则

$$\omega(w_1, \cdots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \cdots, v_n).$$

证 定义 $\eta \in \mathfrak{T}^n(\mathbf{R}^n)$ 如下：

$$\eta((a_{11},\cdots,a_{1n}),\cdots,(a_{n1},\cdots,a_{nn}))$$
$$=\omega\left(\sum a_{1j}v_j,\cdots,\sum a_{nj}v_j\right).$$

显然 $\eta\in\Omega^n(\mathbf{R}^n)$ 所以对某个 $\lambda\in\mathbf{R}$，$\eta=\lambda\cdot\det$，而且

$$\lambda=\eta(e_1,\cdots,e_n)=\omega(v_1,\cdots,v_n).\ \blacksquare$$

定理 **4-6** 表明，一个非零的 $\omega\in\Omega^n(V)$ 可将 V 的基底分为互不相交的两类，一类使 $\omega(v_1,\cdots,v_n)>0$，另一类使 $\omega(v_1,\cdots,v_n)<0$. 若 $v_1,\cdots,v_n$ 与 $w_1,\cdots,w_n$ 是两个基底而矩阵 $A=(a_{ij})$ 由 $w_i=\sum a_{ij}v_j$ 定义，则当且仅当 $\det A>0$ 时，$v_1,\cdots,v_n$ 和 $w_1,\cdots,w_n$ 才属于同一类. 这个判据是与 ω 无关的，所以总可用来把 V 的基底分成互不相交的两类. 每一类称为 V 的一个**定向**. 基底 $v_1,\cdots,v_n$ 所属的定向记作 $[v_1,\cdots,v_n]$，另一个定向则记作 $-[v_1,\cdots,v_n]$. 在 $\mathbf{R}^n$ 中我们定义 $[e_1,\cdots,e_n]$ 为**通常的定向**.

$\dim\Omega^n(\mathbf{R}^n)=1$ 这个式子对于大家可能并不陌生，因为 det 通常定义为 $\omega\in\Omega^n(\mathbf{R}^n)$ 中使 $\omega(e_1,\cdots,e_n)=1$ 的惟一元素. 对于一般的向量空间 V，没有额外的此类判据来判断一个特定元素 $\omega\in\Omega^n(V)$. 然而，设已给了 V 的一个内积 T. 若 $v_1,\cdots,v_n$ 和 $w_1,\cdots,w_n$ 分别是对于 T 的两个标准正交基底，矩阵 $A=(a_{ij})$ 由 $w_i=\sum_{j=1}^n a_{ij}v_j$ 定义，于是

$$\delta_{ij}=T(w_i,w_j)=\sum_{k,l=1}^n a_{ik}a_{jl}T(v_k,v_l)$$
$$=\sum_{k=1}^n a_{ik}a_{jk}.$$

换言之，若以 $A^{\mathrm T}$ 记矩阵 A 的转置，则我们有 $A\cdot A^{\mathrm T}=I$，所以 $\det A=\pm1$. 由定理 4-6 可知，若 $\omega\in\Omega^n(V)$ 满足 $\omega(v_1,\cdots,v_n)=\pm1$，则 $\omega(w_1,\cdots,w_n)=\pm1$. 若已给定了 V 的一个定向 μ，则对任意一个使得 $[v_1,\cdots,v_n]=\mu$ 的标准正交基底 $v_1,\cdots,v_n$，必有惟一的 $\omega\in\Omega^n(V)$ 使 $\omega(v_1,\cdots,v_n)=1$. 这个惟一的 ω 称为由内积 T 和定向 μ 所决定的 V 的**体积元素**. 注意，det 是 $\mathbf{R}^n$ 中由通常内积和通常定向所决定的体积元素，而 $|\det(v_1,\cdots,v_n)|$ 则是由 0 到 $v_1,\cdots,v_n$ 之各个线段所张的

平行多面体的体积.

我们以一个限制在 $\mathbf{R}^n$ 上的作法作为本节的结束. 若 $v_1,\cdots,v_{n-1} \in \mathbf{R}^n$ 而 φ 定义为

$$\varphi(w) = \det\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix},$$

于是 $\varphi \in \Omega^1(\mathbf{R}^n)$; 所以有惟一的 $z \in \mathbf{R}^n$ 使得

$$\langle w,z\rangle = \varphi(w) = \det\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix},$$

这个 z 记作 $v_1 \times \cdots \times v_{n-1}$, 叫做 $v_1,\cdots,v_{n-1}$ 的**叉积**. 下述的性质可以直接由定义得出:

$$\begin{aligned} v_{\sigma(1)} \times \cdots \times v_{\sigma(n-1)} &= \operatorname{sgn}\sigma \cdot v_1 \times \cdots \times v_{n-1}, \\ v_1 \times \cdots \times av_i \times \cdots \times v_{n-1} &= a \cdot (v_1 \times \cdots \times v_{n-1}), \\ v_1 \times \cdots \times (v_i + v'_i) \times \cdots \times v_{n-1} &= v_1 \times \cdots \times v_i \times \cdots \times v_{n-1} \\ &\quad + v_1 \times \cdots \times v'_i \times \cdots \times v_{n-1}. \end{aligned}$$

一个“乘积”依赖于两个以上的因子, 这在数学上不是很常见的. 在两个向量 $v,w \in \mathbf{R}^3$ 的情况下, 我们则得到一个比较看得惯的“积” $v \times w \in \mathbf{R}^3$. 正是这个理由, 有时候主张只在 $\mathbf{R}^3$ 中才能定义叉积.

习题

4-1. * 设 $e_1,\cdots,e_n$ 为 $\mathbf{R}^n$ 的通常基底而 $\varphi_1,\cdots,\varphi_n$ 为其对偶基底.

(a) 证明 $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1},\cdots,e_{i_k}) = 1$. 如果在 $\wedge$ 的定义中没有因子 $(k+l)!/k!\,l!$. 那么上式右边会是什么?

(b) 证明 $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1,\cdots,v_k)$ 是从 $\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$ 中取 $i_1,\cdots,i_k$ 各列所得的 $k \times k$ 阶子式的行列式.

4-2. 若$f:V\to V$是一个线性变换且$\dim V = n$，则$f^*:\Omega^n(V)\to\Omega^n(V)$必是乘以常数$c$. 证明$c=\det f$.

4-3. 若$\omega\in\Omega^n(V)$是由T和μ决定的体积元素，而$w_1,\cdots,w_n\in V$，证明

$$|\omega(w_1,\cdots,w_n)| = \sqrt{\det(g_{ij})},$$

其中$g_{ij} = T(w_i,w_j)$. 提示：若$v_1,\cdots,v_n$是一个标准正交基底，而$w_i = \sum_{j=1}^{n} a_{ij}v_j$，证明$g_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$.

4-4. 若ω是由于T和μ决定的V的体积元素，而$f:\mathbf{R}^n\to V$是使$f^*T = \langle,\rangle$的同构，并且使得$[f(e_1),\cdots,f(e_n)] = \mu$，求证$f^*\omega = \det$.

4-5. 若$c:[0,1]\to(\mathbf{R}^n)^n$是连续的，而且每一个$(c^1(t),\cdots,c^n(t))$都是$\mathbf{R}^n$的基底．证明$[c^1(0),\cdots,c^n(0)] = [c^1(1),\cdots,c^n(1)]$. 提示：考虑$\det\circ c$.

4-6. (a) 若$v\in\mathbf{R}^2$，$v\times$是什么?

(b) 若$v_1,\cdots,v_{n-1}\in\mathbf{R}^n$是线性无关的，证明$[v_1,\cdots,v_{n-1},v_1\times\cdots\times v_{n-1}]$是$\mathbf{R}^n$中的通常的定向．

4-7. 证明每一个非零的$\omega\in\Omega^n(V)$都是由V的某个内积T和V的某个定向μ所决定的体积元素．

4-8. 如$\omega\in\Omega^n(V)$是一个体积元素，用ω来定义“叉积”$v_1\times\cdots\times v_{n-1}$.

4-9.* 导出$\mathbf{R}^3$中叉积的以下性质：

(a) $e_1\times e_1 = 0$　　$e_2\times e_1 = -e_3$　　$e_3\times e_1 = e_2$

$e_1\times e_2 = e_3$　　$e_2\times e_2 = 0$　　$e_3\times e_2 = -e_1$

$e_1\times e_3 = -e_2$　　$e_2\times e_3 = e_1$　　$e_3\times e_3 = 0.$

(b) $v\times w = (v^2w^3 - v^3w^2)e_1 + (v^3w^1 - v^1w^3)e_2 + (v^1w^2 - v^2w^1)e_3.$

(c) $|v\times w| = |v|\cdot|w|\cdot|\sin\theta|$，其中$\theta = \angle(v,w)$.

$\langle v\times w,v\rangle = \langle v\times w,w\rangle = 0.$

(d) $\langle v,w\times z\rangle = \langle w,z\times v\rangle = \langle z,v\times w\rangle$

$v\times\langle w\times z\rangle = \langle v,z\rangle w - \langle v,w\rangle z$

$\langle v\times w\rangle\times z = \langle v,z\rangle w - \langle w,z\rangle v.$

(e) $|v\times w| = \sqrt{\langle v,v\rangle\cdot\langle w,w\rangle - \langle v,w\rangle^2}.$

4-10. 若$w_1,\cdots,w_{n-1}\in\mathbf{R}^n$，求证

$$|w_1\times\cdots\times w_{n-1}| = \sqrt{\det(g_{ij})},$$

其中$g_{ij}=\langle w_i,w_j\rangle$. 提示：对$\mathbf{R}^n$的某个$n-1$维子空间应用习题4-3.

4-11. 若T是V的一个内积，$f:V\to V$是一个线性变换，若对

$x, y \in V, T(x, f(y)) = T(f(x), y)$，$f$ 就称（关于 T）是**自伴**的．若 $v_1, \cdots, v_n$ 是一个标准正交基底而 $A = (a_{ij})$ 是 f 对此基底的矩阵，求证 $a_{ij} = a_{ji}$.

4-12. 若 $f_1, \cdots, f_{n-1}: \mathbf{R}^m \to \mathbf{R}^n$，定义 $f_1 \times \cdots \times f_{n-1}: \mathbf{R}^m \to \mathbf{R}^n$ 为 $f_1 \times \cdots \times f_{n-1}(p) = f_1(p) \times \cdots \times f_{n-1}(p)$. 应用习题2-14 导出 $D(f_1 \times \cdots \times f_{n-1})$ 的一个公式（当 $f_1, \cdots, f_{n-1}$ 为可微时.）

4.2 向量场与微分形式

若 $p \in \mathbf{R}^n$，对于 $v \in \mathbf{R}^n$ 所有的有序偶 (p, v) 之集记作 $\mathbf{R}^n{}_p$，并称之为 $\mathbf{R}^n$ 在 p 的**切空间**．若定义

$$
\begin{aligned}
(p, v) + (p, w) &= (p, v + w), \\
a \cdot (p, v) &= (p, av),
\end{aligned}
$$

这个集极其明显构成一个向量空间．向量 $v \in \mathbf{R}^n$ 时常画成方向为从 0 到 v 的一条带箭头的线段．向量 $(p, v) \in \mathbf{R}^n{}_p$ 可以画成长度、方向与 v 相同的带箭头的线段，但起点在 p（图 4-1），方向从 p 到 $p + v$，所以我们定义 $p + v$ 为 (p, v) 的**终点**．我们通常把 (p, v) 写作 v_p（读作：在 p 点的向量 v).

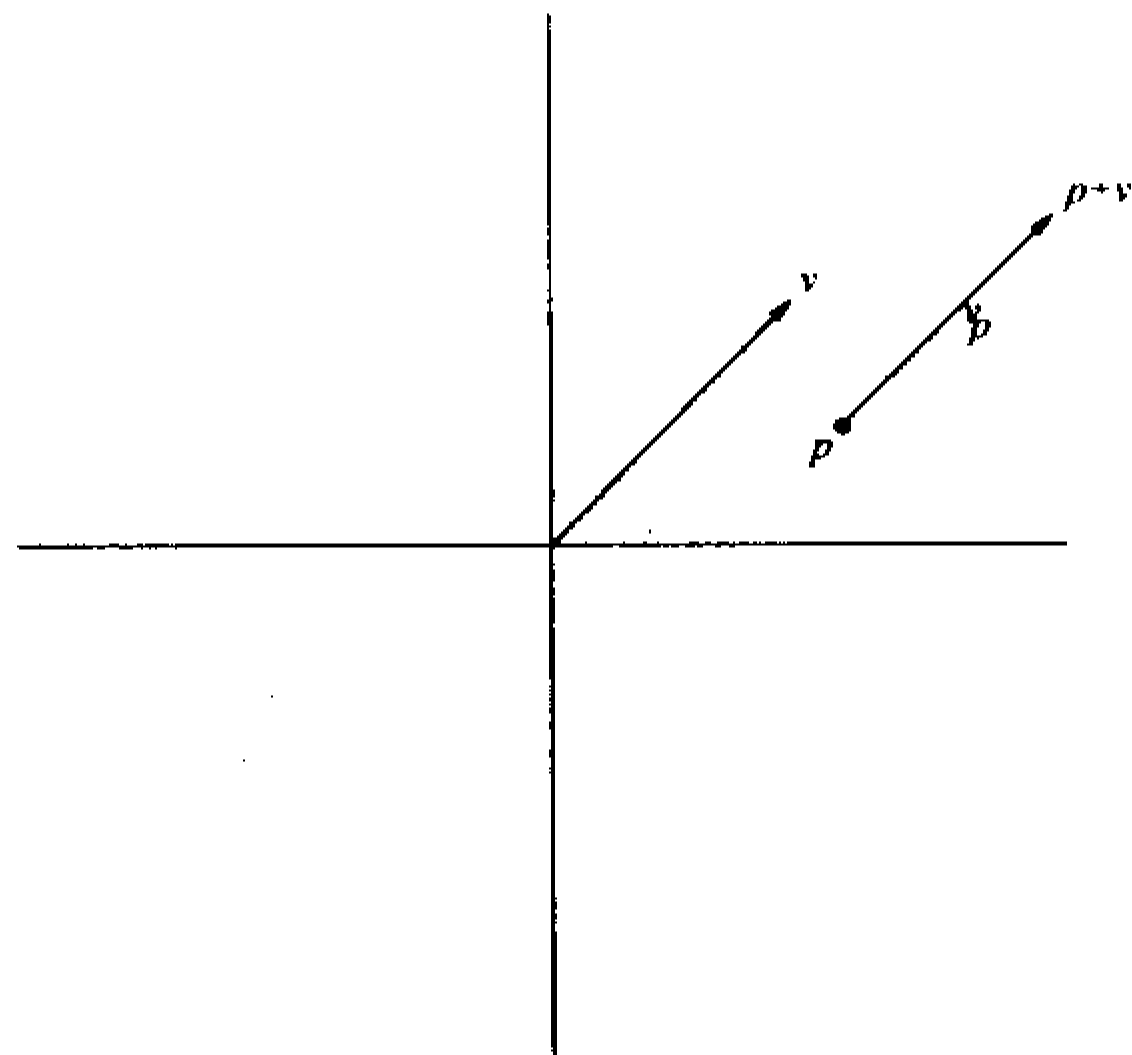


图 4-1

向量空间$\mathbf{R}^n{}_p$与$\mathbf{R}^n$是如此地相近，以致$\mathbf{R}^n$上的许多构造在$\mathbf{R}^n{}_p$上都有类似物．特别是$\mathbf{R}^n_p$中的**通常内积**$\langle\,,\rangle_p$定义为$\langle v_p,w_p\rangle^p=\langle v,w\rangle$，$\mathbf{R}^n{}_p$的**通常定向**则是$[(e_1)_p,\cdots,(e_n)_p]$.

任一个可在向量空间里进行的运算都可以在每个$\mathbf{R}^n{}_p$里实行，这一节的大部分几乎就是把这个话题弄确切．向量空间里最简单的运算就是从中选出一个向量．如果在每一个$\mathbf{R}^n{}_p$里都作这样一个选择，就得到一个**向量场**（图4-2）．确切些说，向量场就是一个函数F，使对每个$p\in\mathbf{R}^n$，$F(p)\in\mathbf{R}^n{}_p$．对每个p都有实数$F^1(p),\cdots,F^n(p)$使

$$F(p)=F^1(p)\cdot(e_1)_p+\cdots+F^n(p)\cdot(e_n)_p.$$

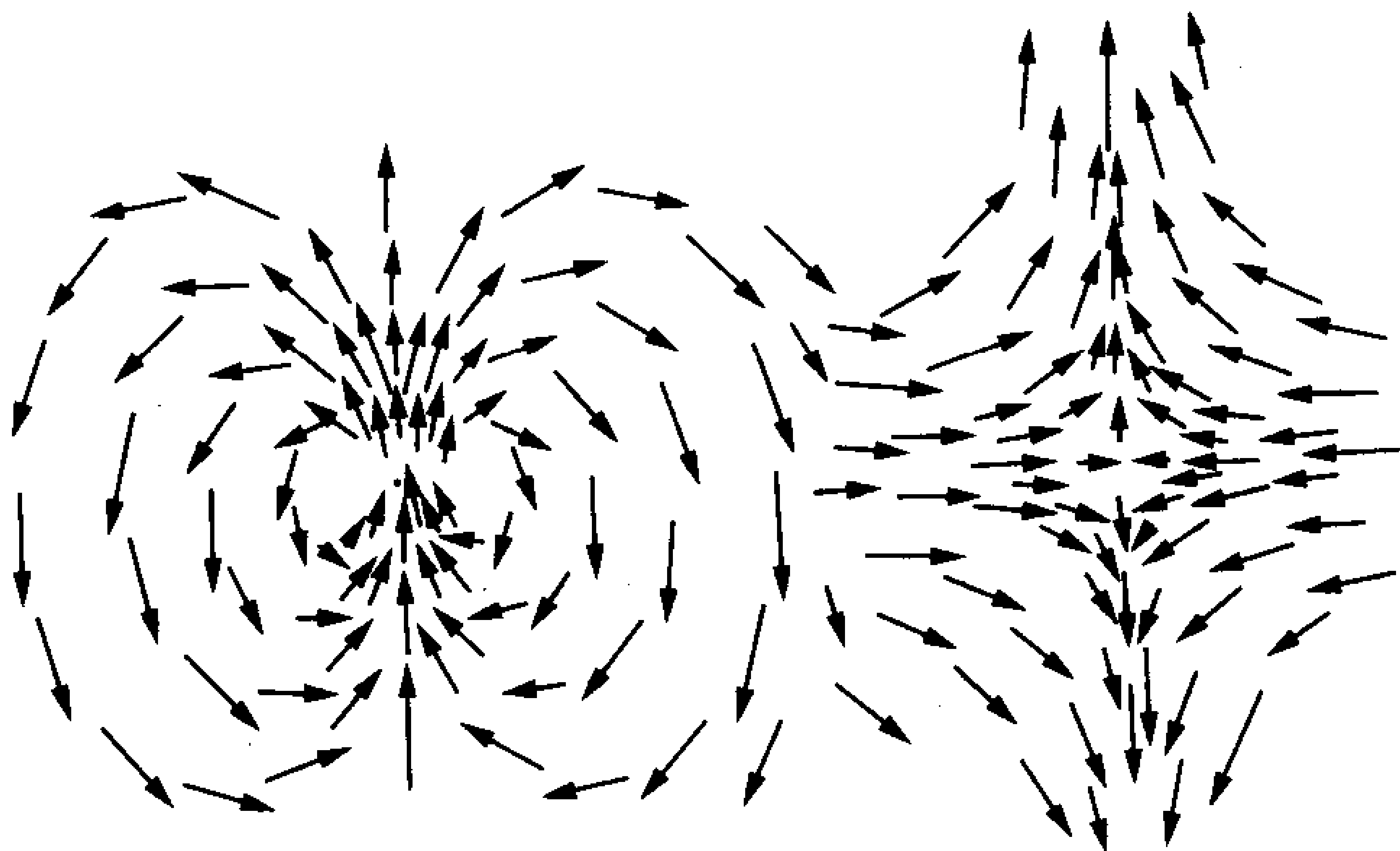


图4-2

于是我们得到n个**分量函数**$F^i:\mathbf{R}^n\to\mathbf{R}$．如果函数$F^i$是连续的、可微的，向量场$F$也就称为连续的、可微的等等．对于只定义在$\mathbf{R}^n$的一个开子集上的向量场，也可作类似的定义．当分别地在每点上作向量运算时，就产生出向量场的运算．例如，如F和G是向量场，而f是一个函数，我们定义

$$(F+G)(p)=F(p)+G(p),$$
$$\langle F,G\rangle(p)=\langle F(p),G(p)\rangle,$$
$$(f\cdot F)(p)=f(p)F(p).$$

如果 $F_1,\cdots,F_{n-1}$ 是 $\mathbf{R}^n$ 上的向量场，那么我们可以类似地定义

$$(F_1\times\cdots\times F_{n-1})(p)=F_1(p)\times\cdots\times F_{n-1}(p).$$

另外一些定义也是标准的和有用的．我们定义 F 的**散度** div F 为 $\sum_{i=1}^{n}\mathrm{D}_iF^i$. 如果引用形式的符号 $\nabla=\sum_{i=1}^{n}\mathrm{D}_i\cdot e_i$，我们可以形式地写作 div $F=\langle\nabla,F\rangle$．若 $n=3$，为与这种符号一致起见，我们写成

$$(\nabla\times F)(p)=(\mathrm{D}_2F^3-\mathrm{D}_3F^2)(e_1)_p+(\mathrm{D}_3F^1-\mathrm{D}_1F^3)(e_2)_p$$
$$+(\mathrm{D}_1F^2-\mathrm{D}_2F^1)(e_3)p.$$

向量场 $\nabla\times F$ 称为 F 的**旋度**，记作 curl F. “散度”和“旋度”的名称都出自物理上的考虑，将在本书末尾加以解释．

对于函数 ω，$\omega(p)\in\Omega^k(\mathbf{R}^n{}_p)$，也可作许多类似的考虑. 这种函数称为 $\mathbf{R}^n$ 上的 k **次形式**，简称**微分形式**．若 $\varphi_1(p),\cdots,\varphi_n(p)$ 是 $(e_1)_p,\cdots,(e_n)_p$ 的对偶基底，则

$$\omega(p)=\sum_{i_1<\cdots<i_k}\omega_{i_1,\cdots,i_k}(p)\cdot[\varphi_{i_1}(p)\wedge\cdots\wedge\varphi_{i_k}(p)].$$

这里 $\omega_{i_1,\cdots,i_k}(p)$ 是某些函数. 如果这些函数是连续的或可微的，就说微分形式 ω 是连续的或可微的等等．我们通常默认微分形式和向量场都是可微的，而“可微”总是指的“C^∞”．这是一个简化的假设，使得以后在证明中无需计算一个函数微分了多少次．和 $\omega+\eta$，积 $f\cdot\omega$. 以及楔积 $\omega\wedge\eta$ 都是按明显的方式定义的. 一个函数 f 看作是 0 次形式，而 $f\cdot\omega$ 也可写作 $f\wedge\omega$.

若 $f:\mathbf{R}^n\to\mathbf{R}$ 是可微的，则 $\mathrm{D}f(p)\in\Omega^1(\mathbf{R}^n)$. 作一点细小的改动，我们就得到一个 1 次形式 $\mathrm{d}f$，定义为：

$$\mathrm{d}f(p)(v_p)=\mathrm{D}f(p)(v).$$

让我们特别考虑一下 1 次形式 $\mathrm{d}\pi^i$. 习惯上是用 x^i 来记函数 π^i.（在 $\mathbf{R}^3$ 上我们时常把 x^1,x^2 和 x^3 记作 x，y 和 z.）这个标准的记号有着明显的缺点，但却使许多经典的结果能用同样经典外表的公式来表示．因为 $\mathrm{d}x^i(p)(v_p)=\mathrm{d}\pi^i(p)(v_p)=\mathrm{D}\pi^i(p)(v)=v^i$，我们看到 $\mathrm{d}x^1(p)$，

$\cdots, dx^n(p)$ 正是 $(e_1)_p, \cdots, (e_n)_p$ 的对偶基底．于是每一个 k 次形式 ω 都可以写成

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

df 的表达式特别有意思．

定理 4-7　若 $f: \mathbf{R}^n \to \mathbf{R}$ 可微，则

$$df = D_1 f \cdot dx^1 + \cdots + D_n f \cdot dx^n.$$

或用经典的记号来写

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n.$$

证　$$df(p)(v_p) = Df(p)(v) = \sum_{i=1}^n v^i \cdot D_i f(p)$$
$$= \sum_{i=1}^n dx^i(p)(v_p) \cdot D_i f(p). \blacksquare$$

现在如果考虑一个可微函数 $f: \mathbf{R}^n \to \mathbf{R}^m$，就有一个线性变换 $Df(p): \mathbf{R}^n \to \mathbf{R}^m$. 因此，再作一点小的修改就得出一个线性变换 $f_*: \mathbf{R}^n{}_p \to \mathbf{R}^m{}_{f(p)}$，定义如下：

$$f_*(v_p) = (Df(p)(v))_{f(p)}.$$

这个线性变换诱导出另一个线性变换 $f^*: \Omega^k(\mathbf{R}^m{}_{f(p)}) \to \Omega^k(\mathbf{R}^n{}_p)$. 若 ω 是 $\mathbf{R}^m$ 上的一个 k 次形式，我们就可以定义 $\mathbf{R}^n$ 上的一个 k 次形式 $f^*\omega$ 为 $(f^*\omega)(p) = f^*(\omega(f(p)))$. 回想一下，这个式子的含义是，对 $v_1, \cdots, v_k \in \mathbf{R}^n{}_p$，我们有 $(f^*\omega)(p)(v_1, \cdots, v_k) = \omega(f(p))(f_*(v_1), \cdots, f_*(v_k))$. 我们现在提出一个定理，它概括了 f^* 的重要性质，用此能直观地计算 $f^*\omega$，作为对这些定义的抽象性的一个矫正．

定理 4-8　若 $f: \mathbf{R}^n \to \mathbf{R}^m$ 是可微的，则

(1) $f^*(dx^i) = \sum_{j=1}^n D_j f^i \cdot dx^j = \sum_{j=1}^n \frac{\partial f^i}{\partial x^j} dx^j$.

(2) $f^*(\omega_1 + \omega_2) = f^*(w_1) + f^*(\omega_2)$.

(3) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$.

(4) $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$.

证

(1) $f^*(\mathrm{d}x^i)(p)(v_p) = \mathrm{d}x^i(f(p))(f_* v_p)$

$= \mathrm{d}x^i(f(p))\left(\sum_{j=1}^n v^j \cdot D_j f^1(p), \cdots, \sum_{j=1}^n v^j \cdot D_j f^m(p)\right)_{f(p)}$

$= \sum_{j=1}^n v^j \cdot D_j f^i(p) = \sum_{j=1}^n D_j f^i(p) \cdot \mathrm{d}x^j(p)(v_p)$.

(2),(3),(4) 留给读者证明. ▌

反复应用定理 4-8 就有例如

$$f^*(P\mathrm{d}x^1 \wedge \mathrm{d}x^2 + Q\mathrm{d}x^2 \wedge \mathrm{d}x^3) = (P \circ f)[f^*(\mathrm{d}x^1) \wedge f^*(\mathrm{d}x^2)] + (Q \circ f)[f^*(\mathrm{d}x^2) \wedge f^*(\mathrm{d}x^3)].$$

把每个 $f^*(\mathrm{d}x^i)$ 都展开是够复杂的了.（然而记住 $\mathrm{d}x^i \wedge \mathrm{d}x^i = (-1)\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$ 会有所帮助.）在一个特殊情况下作一个直观的计算将是值得的.

定理 4-9 若 $f: \mathbf{R}^n \to \mathbf{R}^n$ 是可微的，则

$$f^*(h\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (h \circ f)(\det f')\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

证 因为

$$f^*(h\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (h \circ f)f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n),$$

只要证明

$$f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (\det f')\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

即可. 令 $p \in \mathbf{R}^n$, $A = (a_{ij})$ 为 $f'(p)$ 的矩阵. 今后凡不致引起误解处，为方便计，在 $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(p)$ 等中都略去“p”. 于是由定理 4-6 有

$$\begin{aligned} & f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)(e_1, \cdots, e_n) \\ &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(f_* e_1, \cdots, f_* e_n) \\ &= \mathrm{d}x^1 \wedge \cdots \wedge \left(\sum_{i=1}^n a_{i1} e_i, \cdots, \sum_{i=1}^n a_{in} e_i\right) \\ &= \det(a_{ij}) \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(e_1, \cdots, e_n). \ ▌ \end{aligned}$$

与微分形式有关的一种重要的运算是推广把0次形式变为1次形式的算子d. 若

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\cdots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

我们定义 ω 的**微分** $d\omega$ 为下面的 $k+1$ 次形式:

$$\begin{aligned} d\omega &= \sum_{i_1 < \cdots < i_k} d\omega_{i_1,\cdots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,\cdots,i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

定理 4-10

(1) $d(\omega + \eta) = d\omega + d\eta$.

(2) 若 ω 是一个 k 次形式，η 是一个 l 次形式，则

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(3) $d(d\omega) = 0$. 简单记为 $d^2 = 0$.

(4) 若 ω 是 $\mathbf{R}^m$ 上的 k 次形式，$f:\mathbf{R}^n \to \mathbf{R}^m$ 是可微的，则

$$f^*(d\omega) = d(f^*\omega).$$

证

(1) 留给读者自证.

(2) 当 $\omega = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $\eta = dx^{j_1} \wedge \cdots \wedge dx^{j_l}$ 时公式成立，因为所有各项均为0. 当 ω 是0次形式时，公式很容易验证. 一般的公式可以从(1)以及这两点导出.

(3) 因为

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,\cdots,i_k}) dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

我们有

$$d(d\omega) = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} D_{\alpha,\beta}(\omega_{i_1,\cdots,i_k})\, dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

在这个求和中，下面的两项成对互消:

$$D_{\alpha,\beta}(\omega_{i_1,\cdots,i_k})dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

$$D_{\beta,\alpha}(\omega_{i_1,\cdots,i_k})dx^\alpha \wedge dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

(4) 当 ω 是 0 次形式时，这是清楚的．用归纳法，假设(4)对于 k 次形式成立．只需证明(4)对 $\omega \wedge dx^i$ 这种类型的 $k+1$ 次形式成立即可．我们有

$$\begin{aligned} f^*(d(\omega \wedge dx^i)) &= f^*(d\omega \wedge dx^i + (-1)^k \omega \wedge d(dx^i)) \\ &= f^*(d\omega \wedge dx^i) = f^*(d\omega) \wedge f^*(dx^i) \\ &= d(f^*\omega \wedge f^*(dx^i)) \qquad \text{由(2)和(3)得} \\ &= d(f^*(\omega \wedge dx^i)). \blacksquare \end{aligned}$$

若 $d\omega = 0$ 就称形式 ω 为**闭形式**，若有某个 η 使 $\omega = d\eta$ 就称 ω 为**恰当形式**．定理4-10说明每个恰当形式都是闭的，很自然要问是否每个闭形式也是恰当的．若 ω 是 $\mathbf{R}^2$ 上的1次形式 $P dx + Q dy$，则

$$\begin{aligned} d\omega &= (D_1 P dx + D_2 P dy) \wedge dx + (D_1 Q dx + D_2 Q dy) \wedge dy \\ &= (D_1 Q - D_2 P) dx \wedge dy. \end{aligned}$$

于是，若 $d\omega = 0$，必有 $D_1 Q = D_2 P$. 习题2-21和3-34表明，存在一个0次形式 f 使 $\omega = df = D_1 f dx + D_2 f dy$. 然而如果 ω 只在 $\mathbf{R}^2$ 的一个子集上定义，那么这种函数 f 却可能不存在．定义在 $\mathbf{R}^2 - 0$ 上的微分形式

$$\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

是一个经典的例子．它时常记作 $d\theta$（θ 的定义见习题3-41），因为在集 $\{(x,y): x < 0, \text{或} x \geq 0 \text{而} y \neq 0\}$ 上，θ 是有定义的，在这里它等于 $d\theta$（习题4-21）．然而要注意，θ 不可能连续地定义在整个 $\mathbf{R}^2 - 0$ 上．如果有一个 $f: \mathbf{R}^2 - 0 \to \mathbf{R}$ 存在，使 $\omega = df$，则 $D_1 f = D_1\theta, D_2 f = D_2\theta$，故 $f = \theta + $ 常数，这就证明了这样的 f 不可能存在．

设

$$\omega = \sum_{i=1}^{n} \omega_i \mathrm{d}x^i$$

是 $\mathbf{R}^n$ 上的一个1次形式，而 ω 又等于

$$\mathrm{d}f = \sum_{i=1}^{n} \mathrm{D}_i f \cdot \mathrm{d}x^i.$$

很明显可以假设$f(0)=0$. 和在习题2-35中一样，我们有

$$\begin{aligned} f(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx)\,\mathrm{d}t \\ &= \int_0^1 \sum_{i=1}^{n} \mathrm{D}_i f(tx) \cdot x^i \mathrm{d}t \\ &= \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x^i \mathrm{d}t. \end{aligned}$$

这就启发我们，若已知 ω，要求出f，就要考虑由

$$I\omega(x) = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x^i \mathrm{d}t$$

定义的函数 $I\omega$. 注意，要使 $I\omega$ 的定义有意义，ω 只要定义在一个具有下列性质的开集 $A \subset \mathbf{R}^n$ 上：只要 $x \in A$，由0到 x 的线段就全在 A 内. 这种开集称为对0为**星形的**（图4-3）. 用稍微复杂一点的计算就能证明，（在一个星形开集上）只要 ω 适合必要条件 $\mathrm{d}\omega = 0$，就有 $\omega = \mathrm{d}(I\omega)$. $I\omega$ 的定义和计算一样都可以大大推广.

定理4-11（邦加莱（Poincaré）引理） 若 $A \subset \mathbf{R}^n$ 是对于0的星形开集，则 A 上每个闭形式都是恰当的.

证 我们将定义一个线性变换 I，它能把一个 l 次形式变为一个 $l-1$ 的形式，使得 $I(0)=0$，且对任何形式 ω，有 $\omega = I(\mathrm{d}\omega) + \mathrm{d}(I\omega)$. 由此知，若 $\mathrm{d}\omega = 0$ 则 $\omega = \mathrm{d}(I\omega)$. 令

$$\omega = \sum_{i_1 < \cdots < i_l} \omega_{i_1,\cdots,i_l} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_l}.$$

因为 A 是星形的，我们可以定义

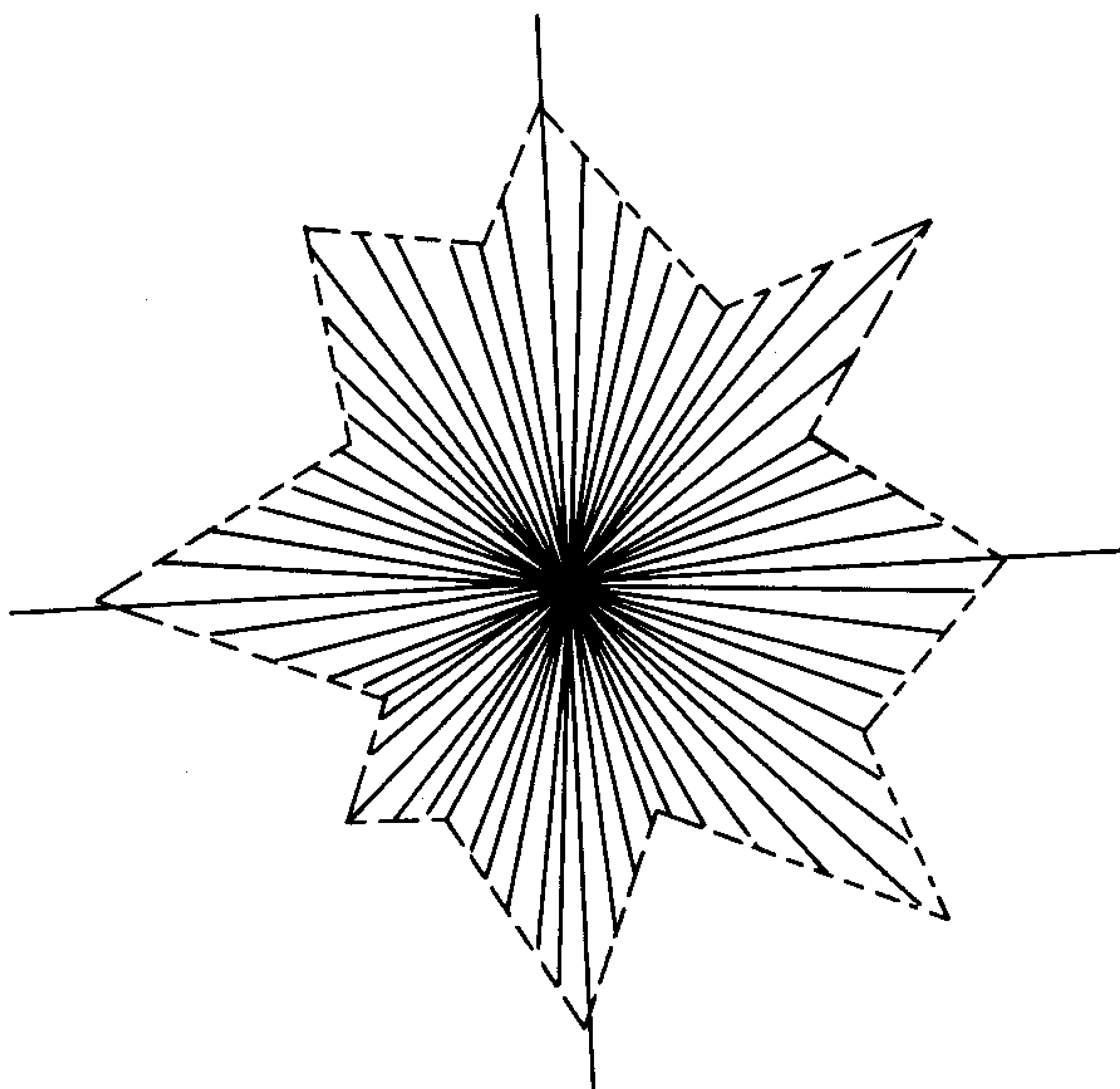


图 4-3

$$I\omega(x) = \sum_{i_1<\cdots<i_l}\sum_{\alpha=1}^{l}(-1)^{\alpha-1}\left(\int_0^1 t^{l-1}\omega_{i_1,\cdots,i_l}(tx)\,\mathrm{d}t\right)x^{i_\alpha}$$

$$\mathrm{d}x^{i_1}\wedge\cdots\wedge\widehat{\mathrm{d}x^{i_\alpha}}\wedge\cdots\wedge\mathrm{d}x^{i_l}.$$

($\mathrm{d}x^{i_\alpha}$ 顶上的符号 $\wedge$ 表示把 $\mathrm{d}x^{i_\alpha}$ 删去.) 要证 $\omega = I(\mathrm{d}\omega) + \mathrm{d}(I\omega)$ 只需作详尽的计算：用习题 3-32，有

$$\mathrm{d}(I\omega)(x) = l\cdot\sum_{i_1<\cdots<i_l}\left(\int_0^1 t^{l-1}\omega_{i_1,\cdots,i_l}(tx)\,\mathrm{d}t\right)\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}$$

$$+\sum_{i_1<\cdots<i_l}\sum_{\alpha=1}^{l}\sum_{j=1}^{n}(-1)^{\alpha-1}\left(\int_0^1 t^{l}\mathrm{D}_j(\omega_{i_1,\cdots,i_l})(tx)\,\mathrm{d}t\right)x^{i_\alpha}$$

$$\mathrm{d}x^{j}\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\widehat{\mathrm{d}x^{i_\alpha}}\wedge\cdots\wedge\mathrm{d}x^{i_l}.$$

(说明何以后项出现因子 t^l 而不是 t^{l-1}.) 我们还有

$$\mathrm{d}\omega = \sum_{i_1<\cdots<i_l}\sum_{j=1}^{n}\mathrm{D}_j(\omega_{i_1,\cdots,i_l})\cdot\mathrm{d}x^{j}\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}.$$

把 I 作用于 $l+1$ 次形式 $\mathrm{d}\omega$，我们有

$$I(\mathrm{d}\omega)(x) = \sum_{i_1<\cdots<i_l}\sum_{j=1}^{n}\left(\int_0^1 t^l \mathrm{D}_j(\omega_{i_1,\cdots,i_l})(tx)\mathrm{d}t\right)x^j\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}$$

$$-\sum_{i_1<\cdots<i_l}\sum_{j=1}^{n}\sum_{\alpha=1}^{l}(-1)^{\alpha-1}\left(\int_0^1 t^l \mathrm{D}_j(\omega_{i_1,\cdots,i_l})(tx)\mathrm{d}t\right)x^{i_\alpha}$$

$$\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\widehat{\mathrm{d}x^{i_\alpha}}\wedge\cdots\wedge\mathrm{d}x^{i_l}.$$

把 $\mathrm{d}(I\omega)(x)$ 和 $I(\mathrm{d}\omega)(x)$ 相加，三重的和式就会消去而得

$$\mathrm{d}(I\omega)+I(\mathrm{d}\omega) = \sum_{i_1<\cdots<i_l} l\cdot\left(\int_0^1 t^{l-1}\omega_{i_1,\cdots,i_l}(tx)\mathrm{d}t\right)\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}$$

$$+\sum_{i_1<\cdots<i_l}\sum_{j=1}^{n}\left(\int_0^1 t^l x^j \mathrm{D}_j(\omega_{i_1,\cdots,i_l})(tx)\mathrm{d}t\right)\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}$$

$$=\sum_{i_1<\cdots<i_l}\left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}[t^l\omega_{i_1,\cdots,i_l}(tx)]\mathrm{d}t\right)\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}$$

$$=\sum_{i_1<\cdots<i_l}\omega_{i_1,\cdots,i_l}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_l}=\omega. \blacksquare$$

习题

4-13. (a) 若 $f:\mathbf{R}^n\to\mathbf{R}^m$ 而 $g:\mathbf{R}^m\to\mathbf{R}^p$，求证 $(g\circ f)_*=g_*\circ f_*$ 而 $(g\circ f)^*=f^*\circ g^*$.

(b) 若 $f,g:\mathbf{R}^n\to\mathbf{R}$，证明 $\mathrm{d}(f\cdot g)=f\cdot\mathrm{d}g+g\cdot\mathrm{d}f$.

4-14. 令 c 为 $\mathbf{R}^n$ 中的可微曲线，即可微函数 $c:[0,1]\to\mathbf{R}^n$. 定义 c 在 t 点的切向量 v 为 $c_*((e_1)_t)=((c^1)'(t),\cdots,(c^n)'(t))_{c(t)}$. 若 $f:\mathbf{R}^n\to\mathbf{R}^m$，证明 $f\circ c$ 在 t 的切向量是 $f_*(v)$.

4-15. 令 $f:\mathbf{R}\to\mathbf{R}$，并定义 $c:\mathbf{R}\to\mathbf{R}^2$ 为 $c(t)=(t,f(t))$. 证明 c 在 t 点的切向量的终点位于 f 之图像在 $(t,f(t))$ 点的切线上.

4-16. 令 $c:[0,1]\to\mathbf{R}^n$ 为一曲线，使对于一切 t 有 $|c(t)|=1$. 证明 $c(t)_{c(t)}$ 与 c 在 t 点的切向量垂直.

4-17. 若 $f:\mathbf{R}^n\to\mathbf{R}^n$，定义向量场 $\mathbf{f}$ 为 $\mathbf{f}(p)=f(p)_p\in\mathbf{R}^n{}_p$.

(a) 证明 $\mathbf{R}^n$ 上每一个向量场 F 都是某个 $\mathbf{f}$ 所生成的 f.

(b) 证明 $\operatorname{div}\mathbf{f}=\operatorname{trace} f'$.

4-18. 若 $f:\mathbf{R}^n\to\mathbf{R}$. 定义向量场 $\operatorname{grad} f$ 为

$$(\text{grad} f)(p) = D_1 f(p)\cdot(e_1)_p + \cdots + D_n f(p)\cdot(e_n)_p.$$

由于明显的理由，我们也写作 grad $f = \nabla f$. 若 $\nabla f(p) = w_p$，证明 $D_v f(p) = \langle v, w\rangle$，并证明 $\nabla f\ (p)$ 的方向是 f 在 p 点变化最快的方向.

4-19. 若 F 是 $\mathbf{R}^3$ 上一个向量场，定义微分形式

$$\omega_F^1 = F^1 dx + F^2 dy + F^3 dz,$$
$$\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy.$$

(a) 求证

$$df = \omega^1_{\text{grad} f},$$
$$d(\omega_F^1) = \omega^2_{\text{curl} F},$$
$$d(\omega_F^2) = (\text{div } F) dx \wedge dy \wedge dz.$$

(b) 用 (a) 求证

$$\text{curl grad} f = 0,$$
$$\text{div curl } F = 0.$$

(c) 若 F 是星形开集 A 上的向量场，而且 curl $F = 0$，证明有某个函数 $f: A\to\mathbf{R}$ 使 $F =$ grad f. 类似地，若 div $F = 0$，证明必有 A 上某向量场 G 使 $F =$ curl G.

4-20. 令 $f: U\to\mathbf{R}^n$ 是可微函数，U 是 $\mathbf{R}^n$ 的开子集，而且有可微逆 $f^{-1}: f(U)\to\mathbf{R}^n$. 假设在 U 上的每个闭形式都是恰当的，求证在 $f(U)$ 上也如此. 提示：若 $d\omega = 0$ 而 $f^*\omega = d\eta$，考虑 $(f^{-1})^*\eta$.

4-21.* 证明在 θ 有定义的集合上恒有

$$d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

4.3 几何预备知识

$A \subset \mathbf{R}^n$ 中的**奇异 n 维立方体**就是一个连续函数 $c: [0,1]^n \to A$（这里 $[0,1]^n$ 表示 n 重乘积 $[0,1]\times\cdots\times[0,1]$）. 用 $\mathbf{R}^0$ 和 $[0,1]^0$ 都表示 $\{0\}$. 于是 A 中一个奇异 0 维立方体就是一个函数 $f: \{0\}\to A$，也就是 A 中一个点. 奇异一维立方体时常叫做**曲线**. 一个特别简单而又

特别重要的 $\mathbf{R}^n$ 中的奇异 n 维立方体的例子是**标准 n 维立方体** $I^n:[0,1]^n \to \mathbf{R}^n$，定义是 $I^n(x)=x, x\in[0,1]^n$.

我们需要考虑 A 中奇异 n 维立方体的整系数形式和，也就是形如

$$2c_1+3c_2-4c_3$$

的表达式，其中，c_1，c_2，c_3 都是 A 中的奇异 n 维立方体．这种具有整系数的奇异 n 维立方体之有限和称为 A 中的一个 **n 维链**[1]．特别是，一个奇异 n 维方体 c 也看作 n 维链 $1\cdot c$．n 维链如何相加和如何乘以整数都是明显的．例如

$$2(c_1+3c_4)+(-2)(c_1+c_3+c_2)=-2c_2-2c_3+6c_4.$$

(习题4-22是这种形式计算的严格叙述．)

对于 A 中每一个奇异 n 维链 c，将定义 A 中一个 $(n-1)$ 维链称为 c 的**边缘**记作 ∂c. 例如 I^2 的边缘可定义为依逆时针方向围着 $[0,1]^2$ 边界的四个奇异一维立方体，如图4-4(a)所示．其实，像图4-4(b)那样，定义 ∂I^2 为这四个奇异一维立方体的具有指定系数的和要方便得多．要给出 ∂I^n 的准确定义需要一些预备性概念．对于 $1\leqslant i\leqslant n$ 的每个 i，我们定义两个奇异 $(n-1)$ 维立方体 $I^n_{(i,0)}, I^n_{(i,1)}$ 如下：如果 $x\in[0,1]^{n-1}$，则

$$\begin{aligned}
I^n_{(i,0)}(x) &= I^n(x^1,\cdots,x^{i-1},0,x^i,\cdots,x^{n-1})\\
&=(x^1,\cdots,x^{i-1},0,x^i,\cdots,x^{n-1}),\\
I^n_{(i,1)}(x) &= I^n(x^1,\cdots,x^{i-1},1,x^i,\cdots,x^{n-1})\\
&=(x^1,\cdots,x^{i-1},1,x^i,\cdots,x^{n-1}).
\end{aligned}$$

我们把 $I^n_{(i,0)}$ 和 $I^n_{(i,1)}$ 分别称为 I^n 的 $(i,0)$ 面和 $I^n_{(i,1)}$ 的 $(i,1)$ 面（图4-5)．然后定义

$$\partial I^n=\sum_{i=1}^{n}\sum_{\alpha=0,1}(-1)^{i+\alpha}I^n_{(i,\alpha)}.$$

对于一般的奇异 n 维立方体 $c:[0,1]^n\to A$ 我们先定义其 (i,α) 面为

1.（或更详细一点，叫**奇异 n 维链**——译者注）

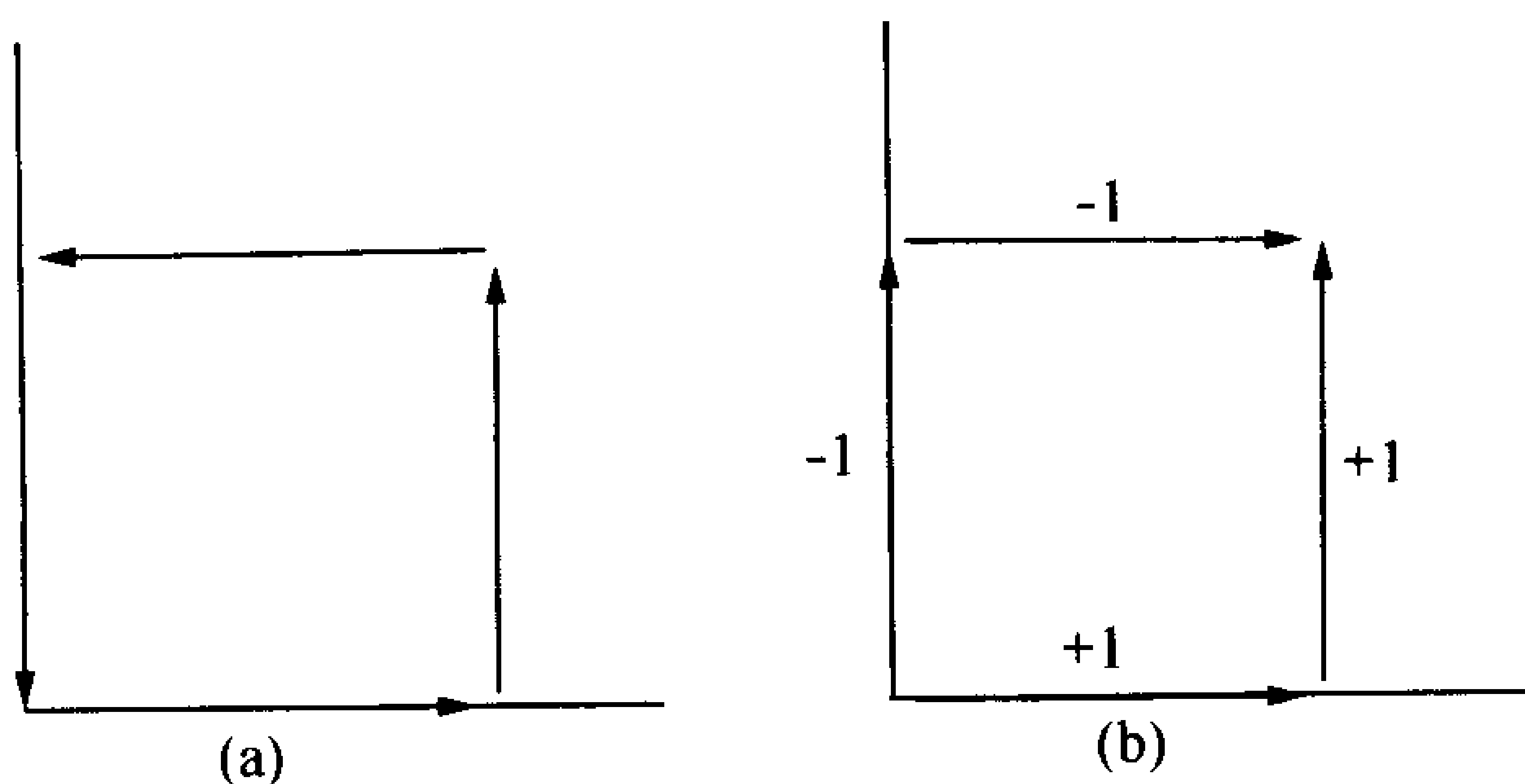


图 4-4

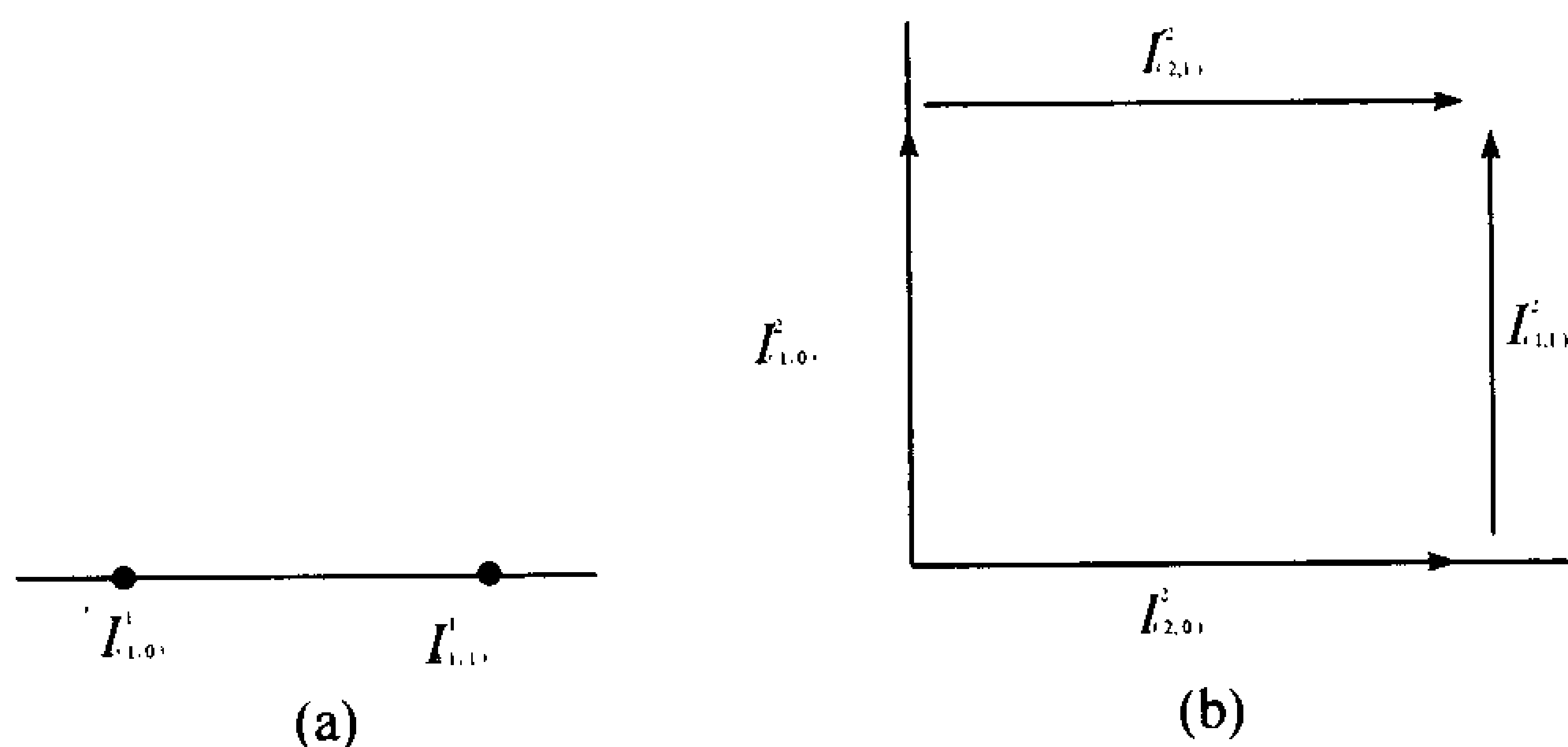


图 4-5

$$c_{(i,\alpha)} = c \circ (I^n_{(i,\alpha)}),$$

再定义

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}.$$

最后我们定义 n 维链 $\sum a_i c_i$ 的边缘为

$$\partial(\sum a_i c_i) = \sum a_i \partial(c_i).$$

虽然这里的少数几个定义对于本书的应用已经够了，我们还是在这里引入 ∂ 的一个标准性质．

定理 4-12 若 c 是 A 中一个 n 维链，则 $\partial(\partial c) = 0$. 简记为，$\partial^2 = 0$.

证　设$i \leqslant j$并考虑$(I^n_{(i,\alpha)})_{(j,\beta)}$. 若$x \in [0,1]^{n-2}$,根据奇异$n$维立方体的$(j,\beta)$面的定义，我们有

$$\begin{aligned}(I^n_{(i,\alpha)})_{(j,\beta)}(x) &= I^n_{(i,\alpha)}(I^{n-1}_{(j,\beta)}(x)) \\ &= I^n_{(i,\alpha)}(x^1,\cdots,x^{j-1},\beta,x^j,\cdots,x^{n-2}) \\ &= I^n(x^1,\cdots,x^{i-1},\alpha,x^i,\cdots,x^{j-1},\beta,x^j,\cdots,x^{n-2}).\end{aligned}$$

类似地有

$$\begin{aligned}(I^n_{(j+1,\beta)})_{(i,\alpha)}(x) &= I^n_{(j+1,\beta)}(I^{n-1}_{(i,\alpha)}(x)) \\ &= I^n_{(j+1,\beta)}(x^1,\cdots,x^{i-1},\alpha,x^i,\cdots,x^{n-2}) \\ &= I^n(x^1,\cdots,x^{i-1},\alpha,x^i,\cdots,x^{j-1},\beta,x^j,\cdots,x^{n-2}).\end{aligned}$$

所以当$i \leqslant j$时$(I^n_{(i,\alpha)})_{(j,\beta)} = (I^n_{(j+1,\beta)})_{(i,\alpha)}$.（在图4-5上检验这一点是有帮助的.）由此易见，对任意奇异n维立方体c，$(c_{(i,\alpha)})_{(j,\beta)} = (c_{(j+1,\beta)})_{(i,\alpha)}$，当$i \leqslant j$. 现在

$$\begin{aligned}\partial(\partial c) &= \partial\Big(\sum_{i=1}^{n}\sum_{\alpha=0,1}(-1)^{i+\alpha}c_{(i,\alpha)}\Big) \\ &= \sum_{i=1}^{n}\sum_{\alpha=0,1}\sum_{j=1}^{n-1}\sum_{\beta=0,1}(-1)^{i+\alpha+j+\beta}(c_{(i,\alpha)})_{(j,\beta)}.\end{aligned}$$

在这个求和中$(c_{(i,\alpha)})_{(j,\beta)}$和$(c_{(j+1,\beta)})_{(i,\alpha)}$的符号相反. 所以所有各项都成对地抵消掉了，而得$\partial(\partial c) = 0$. 因此定理对任意奇异$n$维立方体都成立，它对奇异$n$维链也成立. ▍

很自然会问，定理4-12的逆是否成立：即若$\partial c = 0$，是否有A中的链d存在，使$c = \partial d$？答案与A有关，一般是不行的. 例如，定义$c:[0,1]\to\mathbf{R}^2-0$为$c(t) = (\sin 2\pi nt, \cos 2\pi nt)$，其中$n$是一非零整数. 于是$c(1) = c(0)$而$\partial c = 0$. 但是（习题4-26）不存在$\mathbf{R}^2-0$中的2维链$c'$使$\partial c' = c$.

习题

4-22. 令$\mathcal{S}$表示所有奇异n维立方体之集，$\mathbf{Z}$为整数集. ***n* 维链**就是一个函数$f:\mathcal{S}\to\mathbf{Z}$使除去有限多个$c$以外有$f(c) = 0$, $c \in \mathcal{S}$. 定义$f+g$和$nf(n \in Z)$为$(f+g)(c) = f(c)+g(c)$, $(nf)(c) = n\cdot f(c)$. 证明当f和g是n维链

时，$f+g$ 和 nf 都是 n 维链．若 $c \in \mathcal{S}$，我们又用 c 来表示这样的函数 $f:\mathcal{S}\to\mathbf{Z}$，$f$ 的定义是：$f(c)=1$ 而当 $c'\neq c$ 时 $f(c')=0$. 证明每个 n 维链 f 都可以写作 $a_1c_1+\cdots+a_kc_k$，其中 $a_1,\cdots,a_k$ 是整数，$c_1,\cdots,c_k$ 是奇异 n 维立方体．

4-23. 对于 $R>0$ 和整数 $n\neq0$，定义一个奇异 1 维立方体 $c_{R,n}:[0,1]\to\mathbf{R}^2-0$ 为 $c_{R,n}(t)=(R\cos 2\pi nt,\ R\sin 2\pi nt)$. 证明必有一个奇异 2 维立方体 $c:[0,1]^2\to\mathbf{R}^2-0$ 使 $c_{R_1,n}-c_{R_2,n}=\partial c$.

4-24. 若 c 是 $\mathbf{R}^2-0$ 中的一个奇异 1 维立方体，而且 $c(0)=c(1)$. 证明必有一个整数 n 使 $c-c_{1,n}=\partial c^2$，这里 c^2 是某个 2 维链．提示：先分割 $[0,1]$ 使每一个 $c([t_{i-1},t_i])$ 都包含在通过 0 的某直线的一侧．

4.4 微积分的基本定理

且不说 d 和 ∂ 印刷符号的相似，单看 $d^2=0$ 和 $\partial^2=0$ 这件事，也启发我们看出链和微分形式之间有某种联系．在链上对微分形式作积分就可以建立起这种联系．以后将只考虑可微的奇异 n 维立方体．[1]

若 ω 是 $[0,1]^k$ 上的 k 次形式，则有一个惟一的函数 f 使 $\omega=f\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^k$，我们定义

$$\int_{[0,1]^k}\omega=\int_{[0,1]^k}f.$$

也可以把这个式子写作

$$\int_{[0,1]^k}f\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^k=\int_{[0,1]^k}f(x^1,\cdots,x^k)\,\mathrm{d}x^1\cdots\mathrm{d}x^k,$$

这也是在微分形式的定义中要引入函数 x^i 的理由之一．

如果 ω 是 A 上的 k 次形式而 c 是 A 上的奇异 k 维立方体，我们定义

1. 即指作为奇异 n 维立方体定义的映射 $c:[0,1]^n\to A$ 为可微的．——译者注

$$\int_c \omega = \int_{[0,1]^k} c^* \omega.$$

特别地，注意

$$\int_{I^k} f\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k = \int_{[0,1]^k} (I^k)^* (f\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k)$$
$$= \int_{[0,1]^k} f(x^1,\cdots,x^k)\mathrm{d}x^1\cdots\mathrm{d}x^k.$$

对 $k=0$ 必须给一个特别的定义．一个0次形式 ω 就是一个函数；如果 $c:\{0\}\to A$ 是 A 上的奇异0维立方体，我们定义

$$\int_c \omega = \omega(c(0)).$$

ω 在一个 k 维链 $c = \sum a_i c_i$ 上积分的定义是

$$\int_c \omega = \sum a_i \int_{c_i} \omega.$$

1维链上的一次形式的积分时常称为**线积分**．如果 $P\mathrm{d}x + Q\mathrm{d}y$ 是 $\mathbf{R}^2$ 上的一次形式，而 $c:[0,1]\to\mathbf{R}^2$ 是一个奇异一维立方体(曲线)，我们可以证明(但是不去证它)

$$\int_c P\mathrm{d}x + Q\mathrm{d}y = \lim \sum_{i=1}^n [c^1(t_i) - c^1(t_{i-1})]\cdot P(c(t^i))$$
$$+ [c^2(t_i) - c^2(t_{i-1})]\cdot Q(c(t^i)),$$

其中 $t_0,\cdots,t_n$ 是 $[0,1]$ 的一个分法，在 $[t_{i-1},t_i]$ 中 t^i 的选取是随意的，对所有的分法令最大的 $|t_i - t_{i-1}|$ 趋于0取极限．上式右方时常取作 $\int_c P\mathrm{d}x + Q\mathrm{d}y$ 的定义．因为这些和很像通常积分定义中出现的和，所以作这样的定义是自然的．但是这样一个表达式几乎无法使用，而且很快就发现它等于与 $\int_{[0,1]} c^*(P\mathrm{d}x + Q\mathrm{d}y)$ 等价的一个积分．对**面积分**有类似的定义，即对奇异2维立方体定义二次形式的积分，其定义更复杂，更难于应用．这就是我们避免采用以和作为链上积分之定义这种研究方法的一个理由．另一个理由是，在这里给出的

定义对将在第 5 章考虑的更普遍的情况也有意义.

微分形式、链、d 和 ∂ 之间的关系，以一种可能是最简洁的方式被概括在斯托克斯（Stokes）定理中了，这个定理有时称为高维微积分的基本定理（若 $k=1, c=I^1$，它确实就是微积分的基本定理）.

定理 4-13（斯托克斯定理） 若 ω 是开集 $A\subset\mathbf{R}^n$ 上的一个（$k-1$）次形式而 c 是 A 中 k 维链，则

$$\int_c \mathrm{d}\omega = \int_{\partial c}\omega.$$

证 先设 $c=I^k$ 而 ω 是 $[0,1]^k$ 上的一个$(k-1)$次形式. 于是 ω 是下面类型的$(k-1)$次形式的和

$$f\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^k,$$

只要对每一个这种类型的项证明本定理即可. 这只涉及计算.

注意

$$\begin{aligned}&\int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)}(f\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^k)\\ &=\begin{cases}0 & 若 j\neq i,\\ \int_{[0,1]^k} f(x^1,\cdots,\alpha,\cdots,x^k)\mathrm{d}x^1\cdots\mathrm{d}x^k & 若 j=i.\end{cases}\end{aligned}$$

所以

$$\begin{aligned}&\int_{\partial I^k} f\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^k\\ &=\sum_{j=1}^{k}\sum_{\alpha=0,1}(-1)^{j+\alpha}\int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)}(f\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^k)\\ &=(-1)^{i+1}\int_{[0,1]^k} f(x^1,\cdots,1,\cdots,x^k)\mathrm{d}x^1\cdots\mathrm{d}x^k\\ &\quad+(-1)^i\int_{[0,1]^k} f(x^1,\cdots,0,\cdots,x^k)\mathrm{d}x^1\cdots\mathrm{d}x^k.\end{aligned}$$

另一方面，

$$\int_{I^k}\mathrm{d}(f\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^k)$$

$$= \int_{[0,1]^k} D_i f dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k$$

$$= (-1)^{i-1}\int_{[0,1]^k} D_i f.$$

根据富比尼定理和(一维的)微积分基本定理，有

$$\int_{I^k} d(f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

$$= (-1)^{i-1}\int_0^1 \cdots \left(\int_0^1 D_i f(x^1,\cdots,x^k)dx^i\right)dx^1 \cdots \widehat{dx^i} \cdots dx^k$$

$$= (-1)^{i-1}\int_0^1 \cdots \int_0^1 [f(x^1,\cdots,1,\cdots,x^k)$$

$$-f(x^1,\cdots,0,\cdots,x^k)]dx^1 \cdots \widehat{dx^i} \cdots dx^k$$

$$= (-1)^{i-1}\int_{[0,1]^k} f(x^1,\cdots,1,\cdots,x^k)dx^1 \cdots dx^k$$

$$+ (-1)^{i}\int_{[0,1]^k} f(x^1,\cdots,0,\cdots,x^k)dx^1 \cdots dx^k.$$

所以

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega.$$

若 c 是一个任意的奇异 k 维立方体，按定义算下来可得

$$\int_{\partial c} \omega = \int_{\partial I^k} c^* \omega.$$

所以

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c}\omega.$$

最后，若 c 是一个 k 维链 $\sum a_i c_i$，我们有

$$\int_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int_{\partial c_i} \omega = \int_{\partial c} \omega. \ \blacksquare$$

斯托克斯定理和许多充分展开了重大定理一样，具有三种属性.

1. 它几乎是自明的.
2. 它之所以是自明的，是由于其中出现的名词都作了适当的定义.

3. 它具有重要的推论.

既然整个这一章几乎只是介绍一连串的定义，而正是由于这些定义才有可能叙述和证明斯托克斯定理，所以读者会乐意承认斯托克斯定理的前两点属性. 这本书余下的部分，就将致力于论证这第三个属性.

习题

4-25. (对参数化的独立性) 令 c 为一个奇异 k 维立方体, $p:[0,1]^k \to [0,1]^k$ 是一个 1-1 函数使得 $p([0,1]^k) = [0,1]^k$, 且 $\det p'(x) \geqslant 0, x \in [0,1]^k$. 若 ω 是一个 k 次形式，求证

$$\int_c \omega = \int_{c \circ p} \omega.$$

4-26. 证明 $\int_{c_{R,n}} d\theta = 2\pi n$, 并应用斯托克斯定理证明对任意一个 $\mathbf{R}^2 - 0$ 中的二维链 c, $c_{R,n} \neq \partial c$ ($c_{R,n}$ 的定义见习题 4-23).

4-27. 证明习题 4-24 中的整数 n 是惟一的. 它称为 c 对 0 的**环绕数**.

4-28. 回想一下，复数集 $\mathbf{C}$ 就是 $\mathbf{R}^2$, 但记 $(a,b) = a + bi$. 若 $a_1, \cdots, a_n \in \mathbf{C}$, 令 $f: \mathbf{C} \to \mathbf{C}$ 为 $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. 定义奇异一维立方体 $c_{R,f}: [0,1] \to \mathbf{C} - 0$ 为 $c_{R,f} = f \circ c_{R,1}$, 又令奇异二维立方体 c 为 $c(s,t) = t \cdot c_{R,n}(s) + (1-t) c_{R,f}(s)$.

(a) 求证 $\partial c = c_{R,f} - c_{R,n}$, 又 $c([0,1] \times [0,1]) \subset \mathbf{C} - 0$, 只要 R 充分大.

(b) 用习题 4-26 证明代数基本定理：每个多项式 $z^n + a_1 z^{n-1} + \cdots + a_n, a_i \in \mathbf{C}$, 都在 $\mathbf{C}$ 中有根.

4-29. 若 ω 是 $[0,1]$ 上的一次形式 $f\,dx$, 且 $f(0) = f(1)$, 证明存在惟一数 λ 和某个适合 $g(0) = g(1)$ 的函数 g 使 $\omega - \lambda dx = dg$ 成立. 提示：在 $[0, 1]$ 上对 $\omega - \lambda dx = dg$ 积分以求 λ.

4-30. 若 ω 是 $\mathbf{R}^2 - 0$ 上的一次形式且使得 $d\omega = 0$. 求证必有某 $\lambda \in \mathbf{R}$ 和 $g: \mathbf{R}^2 - 0 \to \mathbf{R}$ 使 $\omega = \lambda d\theta + dg$. 提示：若

$$c_{R,1}^{*}(\omega) = \lambda_R dx + d(g_R),$$

证明所有这样的常数 λ_R 都取相同的值 λ.

4-31. 若 $\omega \neq 0$, 求证必有一个 c 使 $\int_c \omega \neq 0$. 应用这件事、斯托克斯定理和 $\partial^2 = 0$

来证明 $d^2=0$.

4-32. (a) 令 c_1, c_2 是 $\mathbf{R}^2$ 中的奇异一维立方体而且 $c_1(0)=c_2(0)$, $c_1(1)=c_2(1)$. 证明必有一个奇异 2 维立方体 c 使 $\partial c = c_1 - c_2 + c_3 - c_4$, 其中 c_3, c_4 是**退化的**, 即 $c_3([0,1])$ 和 $c_4([0,1])$ 都是点. 由此证明: 若 ω 是恰当的, 则 $\int_{c_1}\omega = \int_{c_2}\omega$. 若 ω 只是闭的, 在 $\mathbf{R}^2-0$ 上给出一个反例.

(b) 若 ω 是 $\mathbf{R}^2$ 的子集上的一次微分形式, 且对所有的奇异一维立方体 c_1, c_2 只要 $c_1(0)=c_2(0)$, $c_1(1)=c_2(1)$ 都有 $\int_{c_1}\omega = \int_{c_2}\omega$, 求证 ω 是恰当的. 提示: 考虑习题 2-21 和 3-34.

4-33. (本题可看作是复变函数论最初步的教程.) 若 $f:\mathbf{C}\to\mathbf{C}$, 而且在 $z_0\in\mathbf{C}$ 处极限

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

存在, 则定义 f 在 z_0 **可微**. (这是两个复数的商, 与第 2 章的定义完全不同.) 若 f 在开集 A 之每一点 z 都可微而且 f' 连续于 A, f 就叫做在 A 上**解析**.

(a) 证明 $f(z)=z$ 解析, $f(z)=\bar{z}$ 则否 ($\overline{x+iy}=x-iy$). 证明解析函数的和、积与商 (在分母不为 0 处) 都是解析的.

(b) 若 $f'=u+iv$ 在 A 上解析, 证明 u 与 v 满足**柯西-黎曼** (Cauchy-Riemann) **方程**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

提示: 利用下述事实: $\lim_{z\to z_0}[f(z)-f(z_0)]/(z-z_0)$ 对于 $z=z_0+(x+i\cdot 0)$ 和 $z=z_0+(0+i\cdot y)$, $x,y\to 0$, 应该相等. (逆定理当 u, v 连续可微时成立, 这比较难证.)

(c) 令 $T:\mathbf{C}\to\mathbf{C}$ 是一个线性变换 ($\mathbf{C}$ 看作是 $\mathbf{R}$ 上的向量空间). 若 T 对于基底 $(1,i)$ 的矩阵是 $\begin{pmatrix} a,b \\ c,d \end{pmatrix}$, 证明当且仅当 $a=d$, $b=-c$ 时, T 才是乘以复数. (b) 证明了任一个解析函数 $f:\mathbf{C}\to\mathbf{C}$, 若看作一个函数 $f:\mathbf{R}^2\to\mathbf{R}^2$, 都有导数 $Df(z_0)$, 它是乘以一个复数的线性变换. 这个复数是什么?

(d) 定义

$$d(\omega + i\eta) = d\omega + i d\eta,$$

$$\int_c \omega + i\eta = \int_c \omega + i\int_c \eta,$$

$$(\omega + i\eta) \wedge (\theta + i\lambda) = \omega \wedge \theta - \eta \wedge \lambda + i(\eta \wedge \theta + \omega \wedge \lambda),$$

以及

$$dz = dx + i dy.$$

证明当且仅当 f 满足柯西-黎曼方程时，$d(f \cdot dz) = 0$.

(e) 证明柯西积分定理：若 f 在 A 上解析，则对每一个闭曲线 c（即 $c(0) = c(1)$ 的奇异一维立方体），只要 A 中有一个 2 维链 c' 使 $c = \partial c'$，必有 $\int_c f dz = 0$.

(f) 证明若 $g(z) = 1/z$，则 $g \cdot dz$ [或用古典的记号写作 $(1/z)dz$] 等于 $i d\theta + dh$，h 是某个函数 $h: \mathbf{C} - 0 \to \mathbf{R}$. 并证：$\int_{c_{R,n}} (1/z)dz = 2\pi i n$.

(g) 若 f 在 $\{z : |z| < 1\}$ 上是解析的，利用 $g(z) = f(z)/z$ 在 $\{z : 0 < |z| < 1\}$ 内解析这一事实，证明

$$\int_{c_{R_1,n}} \frac{f(z)dz}{z} = \int_{c_{R_2,n}} \frac{f(z)}{z}dz,$$

这里 $0 < R_1, R_2 < 1$. 应用 (f) 来计算 $\lim_{R\to 0}\int_{c_{R,n}} [f(z)/z]dz$，并得出结论：

柯西积分公式：若 f 在 $\{z: |z| < 1\}$ 中解析，c 是 $\{z: 0 < |z| < 1\}$ 中的封闭曲线，其对 0 的环绕数是 n，则

$$n \cdot f(0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z}dz.$$

4-34. 若 $F: [0,1] \to \mathbf{R}^3$ 而 $s \in [0,1]$，定义 $F_s: [0,1] \to \mathbf{R}^3$ 为 $F_s(t) = F(s,t)$. 若每一个 F_s 都是封闭曲线，F 称为封闭曲线 F_0 和封闭曲线 F_1 间的一个**同伦**. 假设 F 和 G 是封闭曲线之间的同伦，若对每一个 s，封闭曲线 F_s 和 G_s 都不相交，(F,G) 称为不相交封闭曲线 F_0, G_0 和 F_1, G_1 间的一个同伦. 直观上很明显，若 F_0，G_0 是图 4-6(a) 上的那一对曲线，而 F_1, G_1 是 (b) 或 (c) 上的那一对，其间不可能有同伦. 本题和习题 5-33，对于 (b) 证明了这一点，但对 (c) 的证明需要不同的技巧.

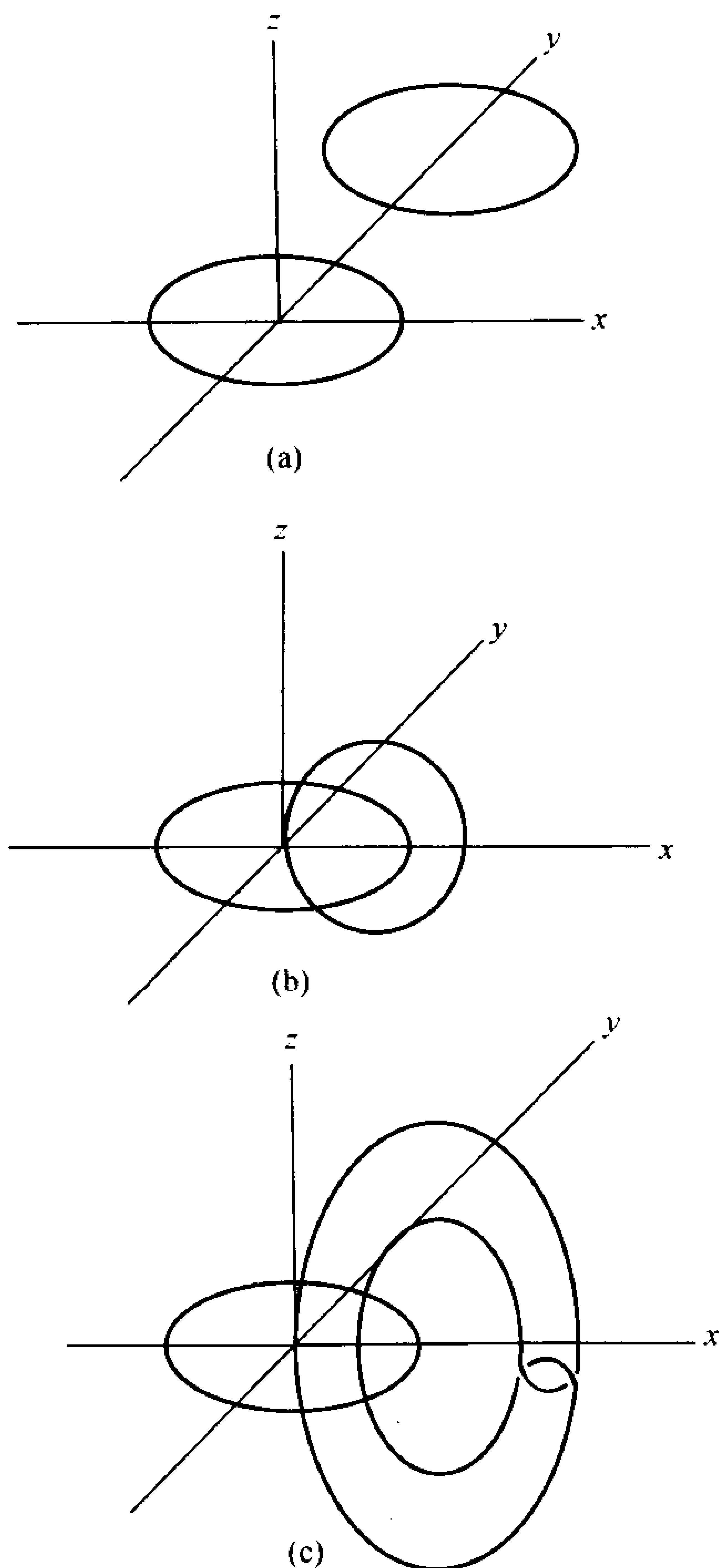


图 4-6

（a）若$f, g:[0,1] \to \mathbf{R}^3$是不相交的封闭曲线，定义$c_{f,g}:[0,1]^2 \to \mathbf{R}^3 - 0$为

$$c_{f,g}(u, v) = f(u) - g(v).$$

若(F,G)是不相交封闭曲线的同伦，定义$C_{F,G}:[0,1]^3 \to \mathbf{R}^3 - 0$为

$$C_{F,G}(s,u,v) = c_{F_s,G_s}(u,v) = F(s,u) - G(s,v).$$

证明

$$\partial C_{F,G} = c_{F_0,G_0} - c_{F_1,G_1}.$$

(b) 若 ω 是 $\mathbf{R}^3-0$ 上的闭二次形式，证明

$$\int_{c_{F_0,G_0}} \omega = \int_{c_{F_1,G_1}} \omega.$$

第5章　流形上的积分

5.1　流　　形

若U和V是$\mathbf{R}^n$中的开集，一个可微函数$h:U\to V$若有可微逆h^{-1}: $V\to U$, 就称为**微分同胚**.（以下“可微”都意味着“C^∞”.）

$\mathbf{R}^n$的子集M，若其每一点$x\in M$都满足下面条件(M)，就称为***k*维流形**.

（M）存在一个含x的开集U，一个开集$V\subset\mathbf{R}^n$和一个**微分同胚**$h:U\to V$使得

$$\begin{aligned} h(U\cap M) &= V\cap(\mathbf{R}^k\times\{0\}) \\ &= \{y\in V: y^{k+1}=\cdots=y^n=0\}. \end{aligned}$$

换句话说, $U\cap M$除了“相差一个微分同胚”，就是$\mathbf{R}^k\times\{0\}$（图5-1）. 在我们的定义中有两个极端的情况应该注意：$\mathbf{R}^n$中的一点是一个0维流形，$\mathbf{R}^n$的一个开子集是一个n维流形.

n维流形的一个常见的例子是***n*维球面**S^n，其定义是$\{x\in\mathbf{R}^{n+1}: |x|=1\}$. 作为一个练习我们留给读者证明其条件(M)成立. 如果你不愿在细节上费神，就可以代之以下面的定理，它可以给出流形的许多例子(注意$S^n=g^{-1}(0)$，其中$g:\mathbf{R}^{n+1}\to\mathbf{R}$定义为$g(x)=|x|^2-1$).

定理5-1　令$A\subset\mathbf{R}^n$为开集而$g:A\to\mathbf{R}^p$是一个可微函数，而且当$g(x)=0$时, $g'(x)$之秩为p. 这时$g^{-1}(0)$是$\mathbf{R}^n$中的一个$(n-p)$维流形.

证　由定理2-13立即可得. ▌

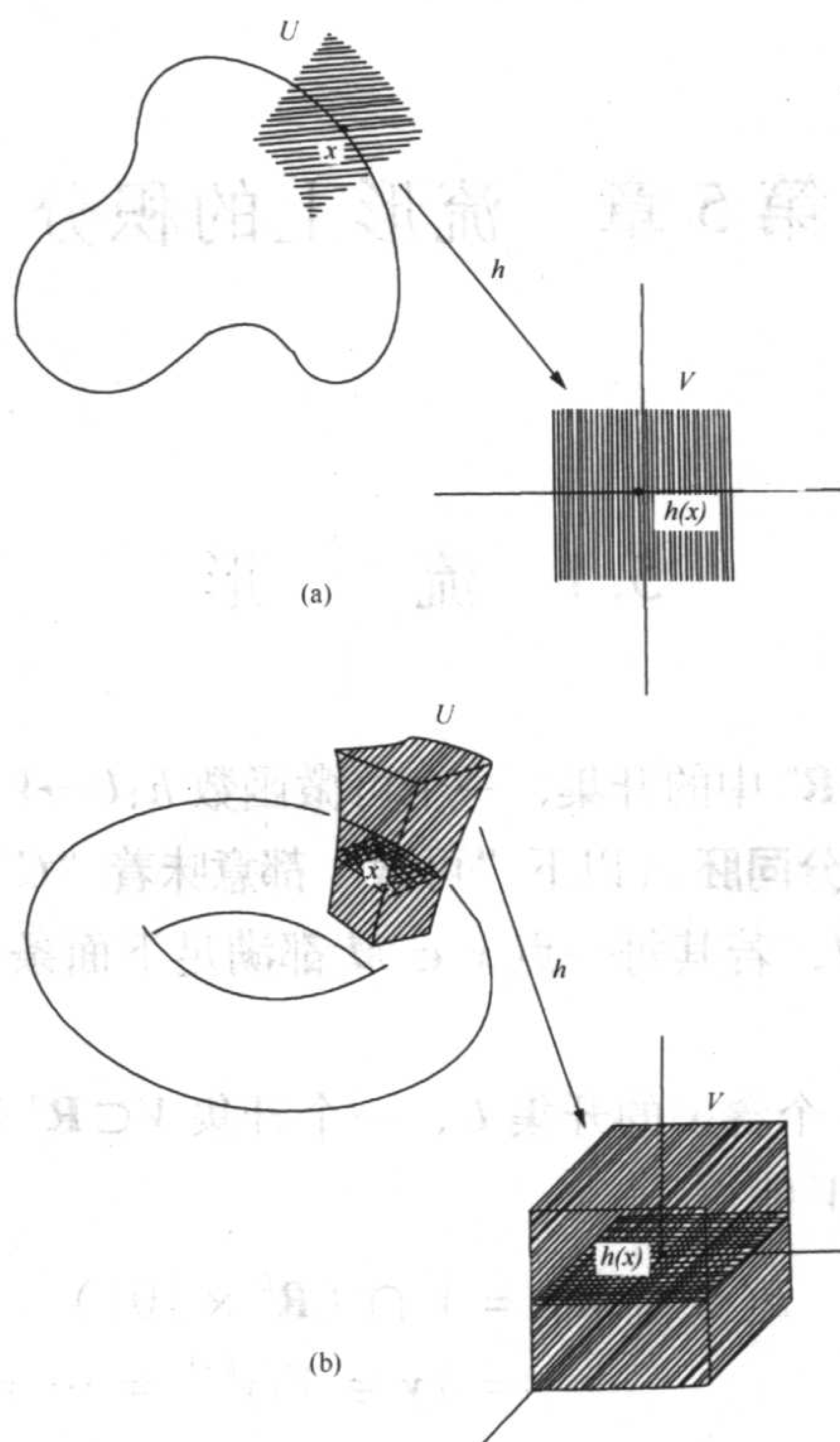


图 5-1　$\mathbf{R}^2$ 中的一个一维流形和 $\mathbf{R}^3$ 中的一个二维流形

流形还有另一种刻画的方法，它是非常重要的.

定理 5-2　$\mathbf{R}^n$ 的子集 M 是 k 维流形，当且仅当对每一点 $x \in M$，下述"坐标条件"成立：

(C) 存在一个包含 x 的开集 U，一个开集 $W \subset \mathbf{R}^k$ 以及一个 1-1 可微函数 $f: W \to \mathbf{R}^n$ 使得：

(1) $f(W) = M \cap U$，

(2) $f'(y)$ 对每个 $y \in W$ 的秩为 k，

(3) $f^{-1}: f(W) \to W$ 是连续的.

[这样一个函数 f 称为 x 周围的**坐标系**（见图 5-2）.]

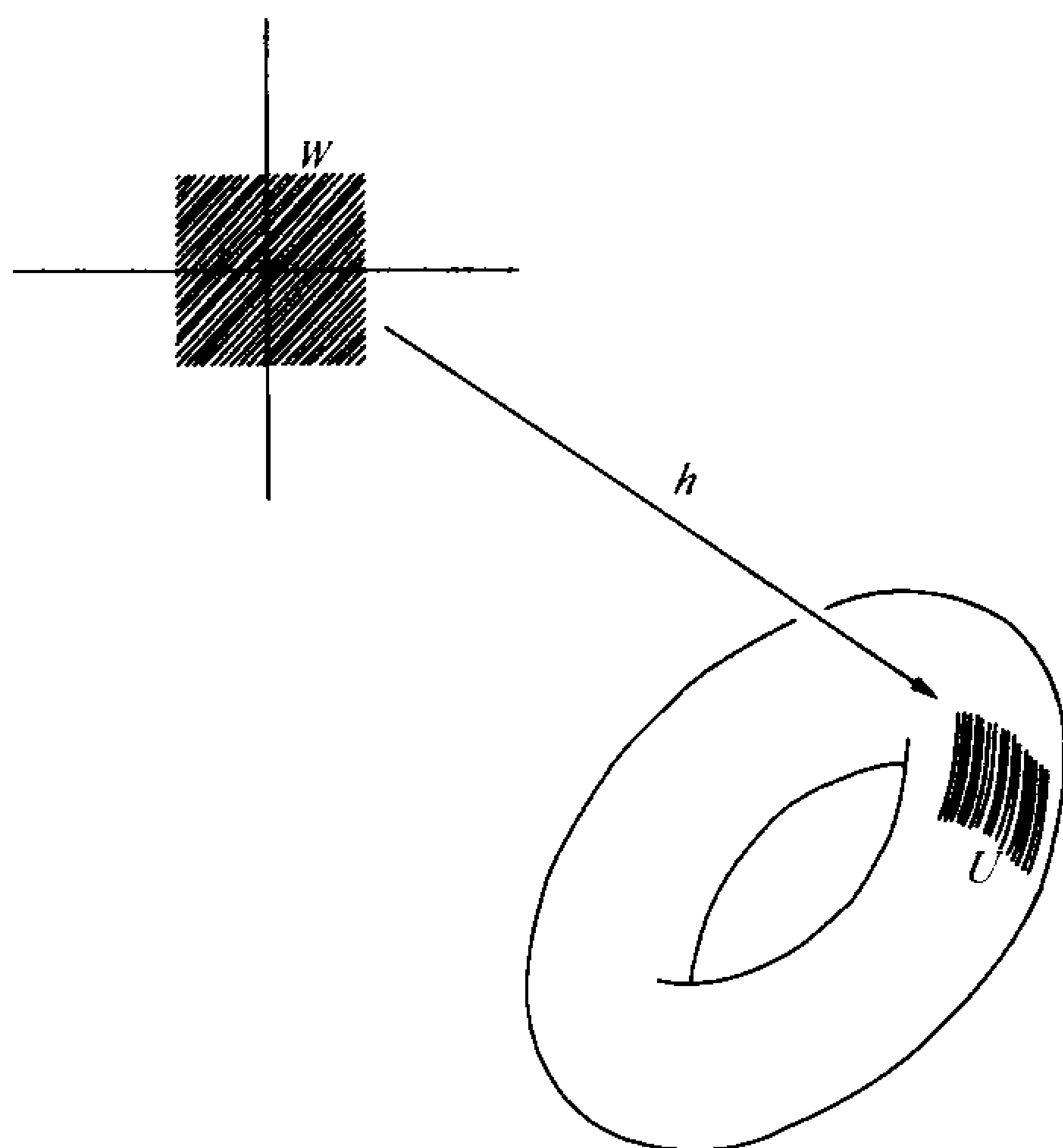


图 5-2

证 若 M 是 $\mathbf{R}^n$ 中的一个 k 维流形，选取 $h:U\to V$ 适合条件(M). 令 $W=\{a\in\mathbf{R}^k:(a,0)\in h(U\cap M)\}$ 并定义 $f:W\to\mathbf{R}^n$ 为 $f(a)=h^{-1}(a,0)$. 显然 $f(W)=M\cap U$，而且 f^{-1} 连续. 如果 $H:U\to\mathbf{R}^k$ 是 $H(z)=(h^1(z),\cdots,h^k(z))$ 则对一切 $y\in W, H(f(y))=y$，所以 $H'(f(y))\cdot f'(y)=I$. 故 $f'(y)$ 的秩必为 k.

反过来，设 $f:W\to\mathbf{R}^n$ 适合条件(C). 令 $x=f(y)$，显然我们可以假设矩阵 $(D_jf^i(y)),1\leqslant i,j\leqslant k$，具有非零行列式. 定义 $g:W\times\mathbf{R}^{n-k}\to\mathbf{R}^n$ 为 $g(a,b)=f(a)+(0,b)$，则 $\det g'(a,b)=\det(D_jf^i(a))$，所以 $\det g'(y,0)\neq 0$. 由定理 2-11，必有一个含 $(y,0)$ 的开集 V_1' 和一个含 $g(y,0)=x$ 的开集 V_2' 使得 $g:V_1'\to V_2'$ 具有可微逆 $h:V_2'\to V_1'$. 因为 f^{-1} 是连续的，$\{f(a):(a,0)\in V_1'\}=U\cap f(W)$，$U$ 是某开集. 令 $V_2=V_2'\cap U, V_1=g^{-1}(V_2)$. 于是 $V_2\cap M$ 恰好是 $\{f(a):(a,0)\in V_1\}=\{g(a,0):(a,0)\in V_1\}$，所以

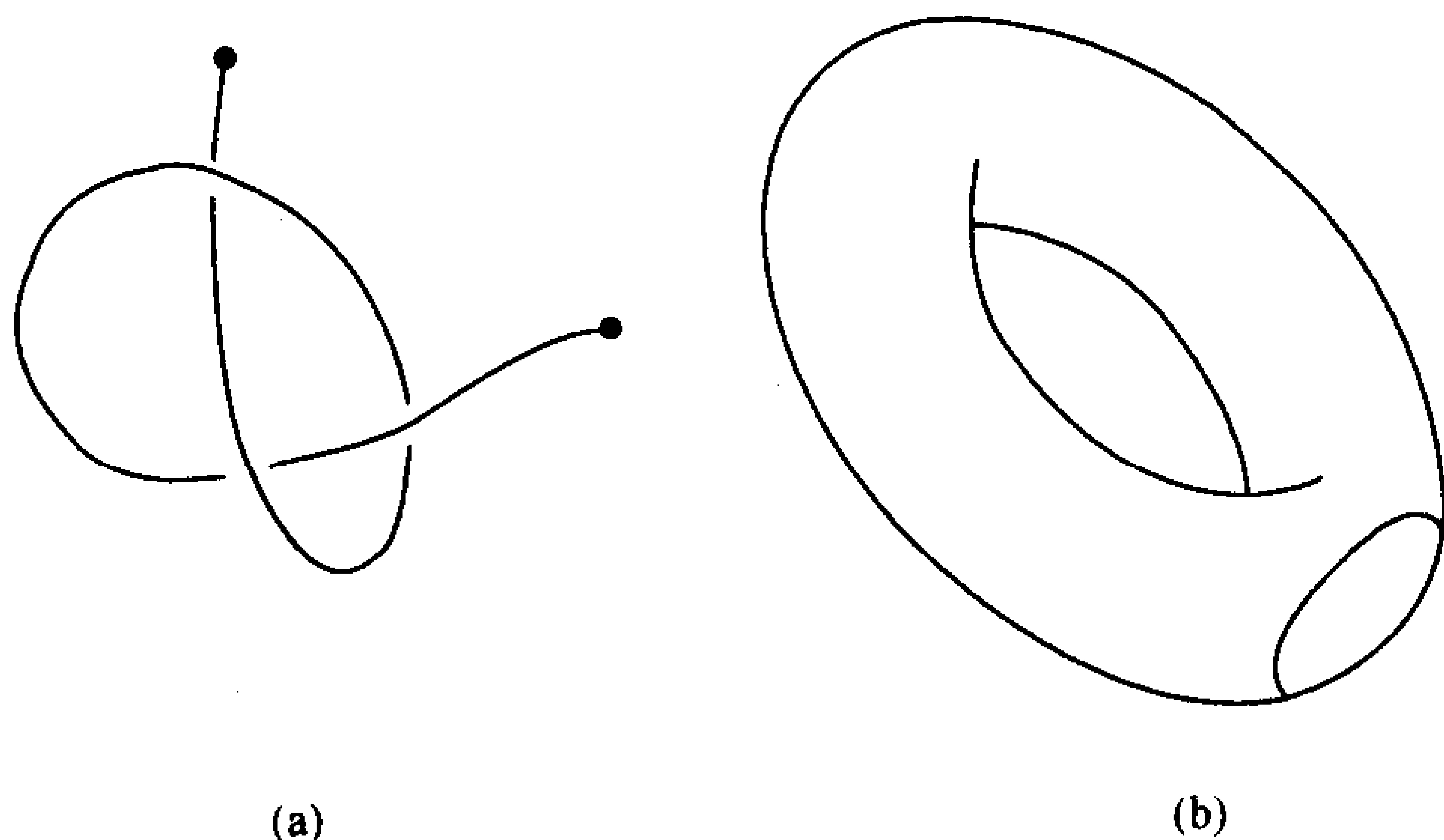
$$\begin{aligned}h(V_2\cap M)&=g^{-1}(V_2\cap M)=g^{-1}(\{g(a,0):(a,0)\in V_1\})\\&=V_1\cap(\mathbf{R}^k\times\{0\}).\ \blacksquare\end{aligned}$$

定理 5-2 的证明有一个推论值得注意．若 $f_1:W_1 \to \mathbf{R}^n$ 和 $f_2:W_2 \to \mathbf{R}^n$ 是两个坐标系，则

$$f_2^{-1}\circ f_1 : f_1^{-1}(f_2(W_2)) \to \mathbf{R}^k$$

是可微的并有非零的雅可比行列式．事实上，$f_2^{-1}(y)$ 就是由 $h(y)$ 的前 k 个分量组成的．

半空间 $\mathbf{H}^k \subset \mathbf{R}^k$ 定义为 $\{x \in \mathbf{R}^k : x^k \geqslant 0\}$．$\mathbf{R}^n$ 的子集 M，若对每一点 $x \in M$，或适合条件(M)或适合下面条件(M′)，就称为 k **维有边流形**(见图 5-3)：



(a)　　(b)

图 5-3　$\mathbf{R}^3$ 中一个一维和一个二维的有边流形

(M′) 存在一个含 x 的开集 U，一个开集 $V \subset \mathbf{R}^n$ 以及一个微分同胚 $h:U \to V$，使得：

$$\begin{aligned} h(U \cap M) &= V \cap (\mathbf{H}^k \times \{0\}) \\ &= \{y \in V : y^k \geqslant 0 \text{ 且 } y^{k+1} = \cdots = y^n = 0\}, \end{aligned}$$

且 $h(x)$ 的第 k 个分量等于 0.

重要的是要注意，对于同一点 x，条件(M)和(M′)不可能同时成立．事实上，若 $h_1:U_1 \to V_1$ 和 $h_2:U_2 \to V_2$ 分别适合条件(M)和(M′)，则 $h_2 \circ h_1^{-1}$ 应该是一个可微映射，它把 $\mathbf{R}^k$ 中含 $h(x)$ 的一个开集变成 $\mathbf{H}^k$ 的一个子集，而后者在 $\mathbf{R}^k$ 中不是开集．因为 $\det(h_2 \circ$

$h_1^{-1})' \neq 0$，这与习题2-36 矛盾．M 中所有适合条件(M′)的点 x 之集合称为 M 的**边缘**，记作 ∂M. 一定不要把它和第 1 章所定义的一个集合的边界弄混了(见习题5-3 和5-8).

习题

5-1. 若 M 是一个 k 维有边流形，证明 ∂M 是一个$(k-1)$维流形，而 $M-\partial M$ 是 k 维流形．

5-2. 当略去条件(3)时给定理5-2 找一个反例．提示：将一个开区间弯成一个"6"字形．

5-3. (a) 设 $A\subset\mathbf{R}^n$ 是一个开集，其边界是一个$(n-1)$维流形．证明 $N=A\cup$(A之边界）是一个 n 维有边流形．（最好记住下一例子：若 $A=\{x\in\mathbf{R}^n: |x|<1$ 或 $1<|x|<2\}$，这时 $N=A\cup$(A 的边界）是一个有边流形，但 $\partial N \neq A$ 的边界．）

(b) 对于 n 维流形的开子集证明类似结论．

5-4. 证明定理5-1 的一个部分的逆：若 $M\subset\mathbf{R}^n$ 是一个 k 维流形且 $x\in M$，则必有包含 x 的一个开集 $A\subset\mathbf{R}^n$ 和一个可微函数 $g:A\to\mathbf{R}^{n-k}$ 使 $A\cap M=g^{-1}(0)$，而且当 $g(y)=0$ 时 $g'(y)$ 的秩是 $n-k$.

5-5. 证明 $\mathbf{R}^n$ 的 k 维（向量）子空间是 k 维流形．

5-6. 若 $f:\mathbf{R}^n\to\mathbf{R}^m$，$f$ 的**图像**是$\{(x,y):y=f(x)\}$．证明当且仅当 f 为可微时，f 的图像是一个 n 维流形．

5-7. 令 $\mathbf{K}^n=\{x\in\mathbf{R}^n:x^1=0,x^2,\cdots,x^{n-1}>0\}$．若 $M\subset\mathbf{K}^n$ 是一个 k 维流形而 N 是由 M 绕轴 $x^1=\cdots=x^{n-1}=0$ 旋转而得的，证明 N 是$(k+1)$维流形．例：环面(图5-4).

5-8. (a) 若 M 是 $\mathbf{R}^n$ 中的 k 维流形，$k<n$，证明 M 具有测度0.

(b) 若 M 是 $\mathbf{R}^n$ 中闭的[1] n 维有边流形，证明 M 的边界就是 ∂M. 若 M 不是闭的，给出一个反例．

(c) 若 M 是 $\mathbf{R}^n$ 中一个紧的 n 维有边流形，证明 M 为约当可测．

1. 这里所谓"闭"是指 M 作为 $\mathbf{R}^n$ 的子集是闭集合．——译者注

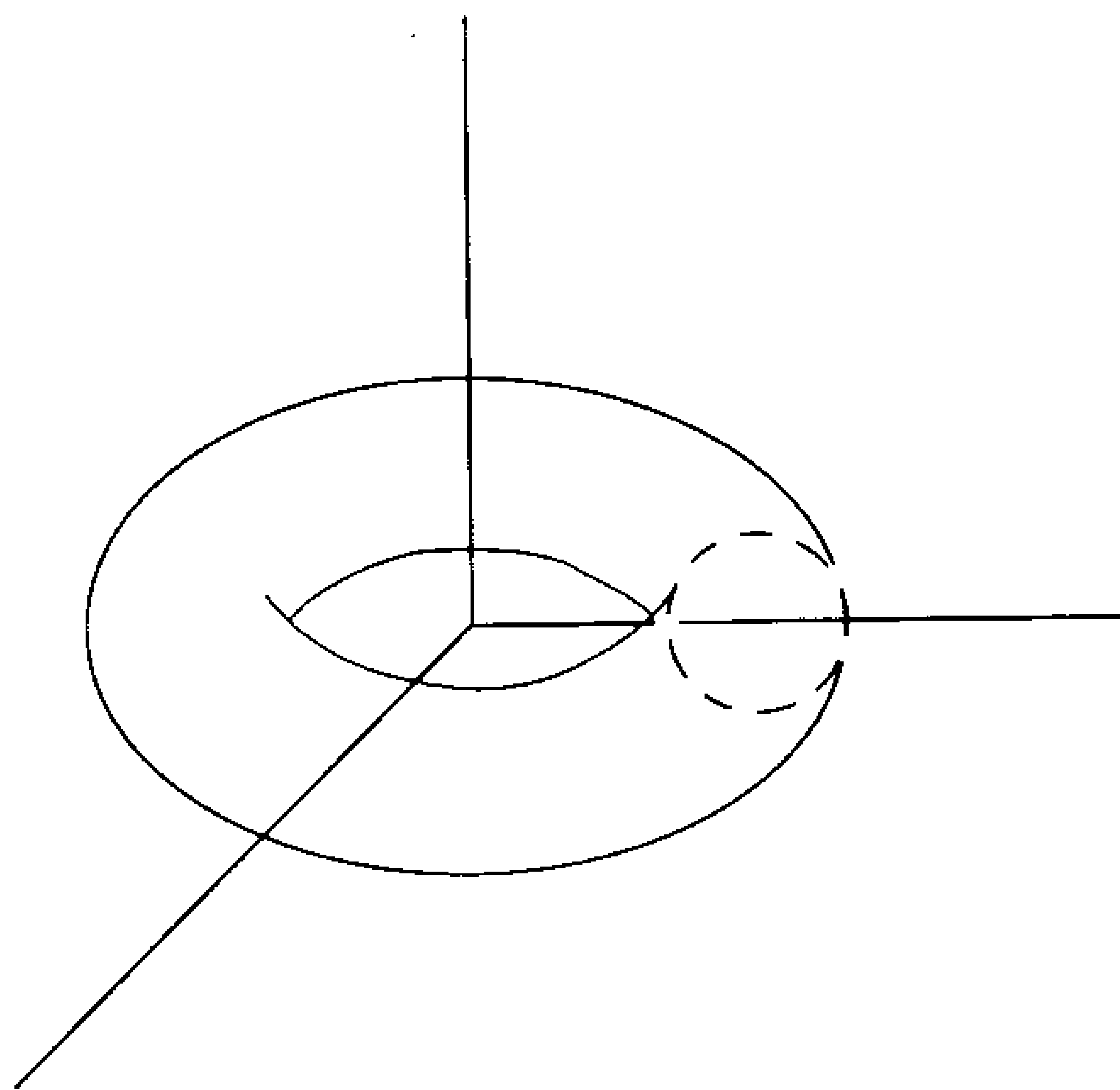


图5-4

5.2　流形上的向量场和微分形式

设 M 为 $\mathbf{R}^n$ 中的一个 k 维流形，$f: W \to \mathbf{R}^n$ 是在 $x = f(a)$ 点周围的一个坐标系．因为 $f'(a)$ 的秩为 k，所以线性变换 $f_*: \mathbf{R}^k_a \to \mathbf{R}^n_x$ 是1-1的，而 $f_*(\mathbf{R}^k_a)$ 是 $\mathbf{R}^n_x$ 的 k 维子空间．若 $g: V \to \mathbf{R}^n$ 是另一个坐标系，$x = g(\mathrm{b})$，则

$$g_*(\mathbf{R}^k_b) = f_*(f^{-1} \circ g)_*(\mathbf{R}^k_b) = f_*(\mathbf{R}^k_a).$$

于是 k 维子空间 $f_*(\mathbf{R}^k_a)$ 并不依赖于坐标系 f．这个子空间记作 M_x，称为 M 在 x 点的**切空间**（见图5-5）．在后面各节里，我们要用到下面这个事实：在 M_x 上有一个由 $\mathbf{R}^n_x$ 中的内积所诱导的自然的内积如下：若 $v, w \in M_x$，定义 $T_x(v,w) = \langle v,w \rangle_x$.

设 A 是一个包含 M 的开集，F 是 A 上的一个可微向量场，且对每个 $x \in M$，$F(x) \in M_x$．若 $f: W \to \mathbf{R}^n$ 是一个坐标系，必有 W 上惟一的（可微）向量场 G 使 $f_*(G(a)) = F(f(a))$，对每一 $a \in W$．我们也可以考虑一个函数 F，它对每个 $x \in M$ 指定一个向量 $F(x) \in M_x$，这样的函数叫做 **M 上的向量场**．W 上仍然有一个惟一的向量场 G 使对 $a \in W$ 有 $f_*(G(a)) = F(f(a))$．如果 G 是可微的，我们就定义 F 是可

微的[1]. 要注意，我们的定义并不依赖于**坐标系的选择**：如果 $g:V\to\mathbf{R}^n$ 是另一个坐标系，并有 V 上惟一向量场 H 使 $g_*(H(b)) = F(g(b))$ 对所有 $b\in V$ 成立，则 $H(b)$ 的分量函数一定等于 $G(f^{-1}(g(b)))$，所以当 G 可微时 H 也是可微的.

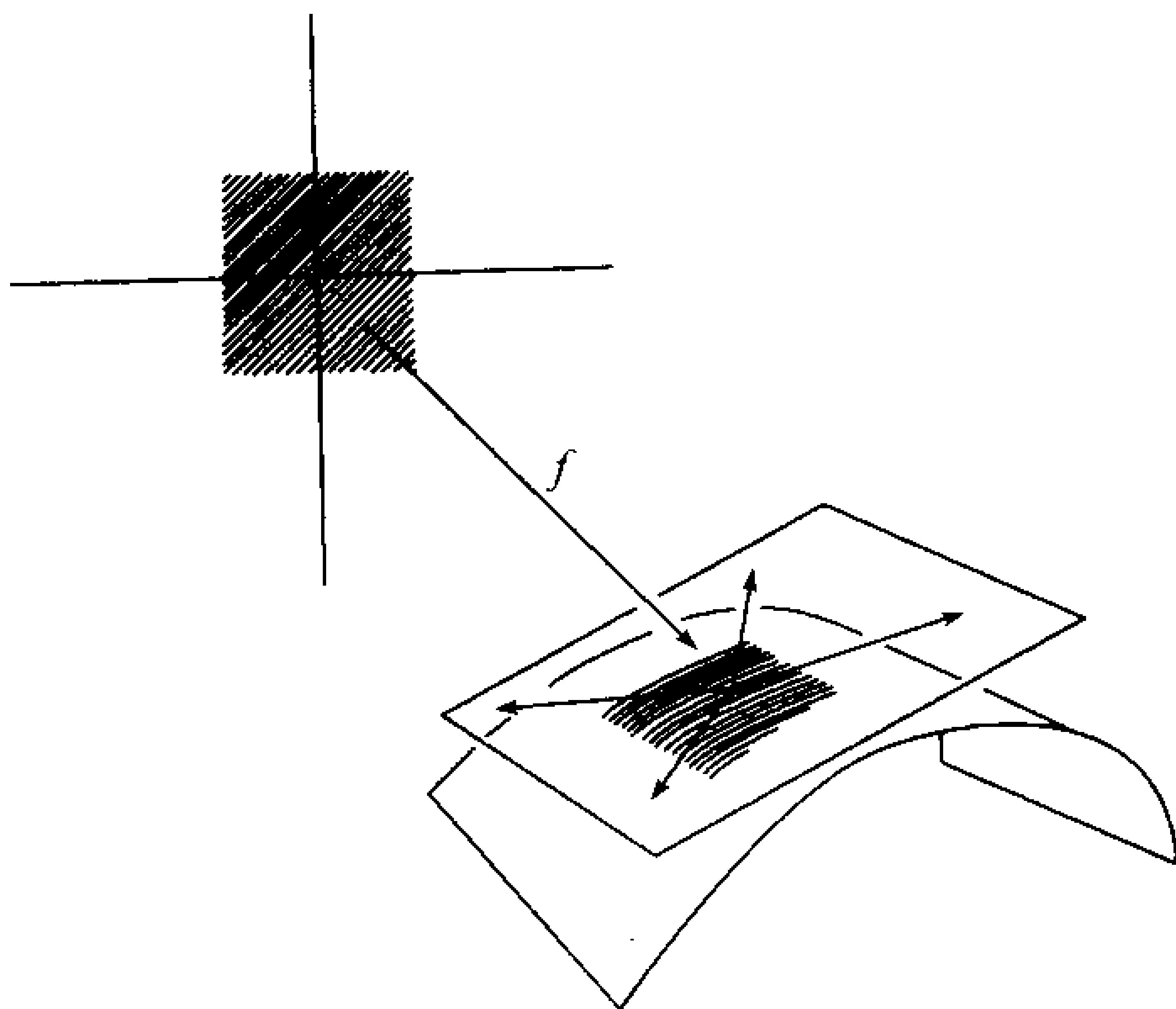


图 5-5

对于微分形式 ω，可以作完全相同的考虑. 一个函数 ω 若对每个 $x\in M$ 都指定一个 $\omega(x)\in\Omega^p(M_x)$，就叫做 **$M$ 上的 p 次形式**. 如果 $f:W\to\mathbf{R}^n$ 是一个坐标系，则 $f^*\omega$ 是 W 上的一个 p 次形式. 如果 $f^*\omega$ 是可微的，我们就定义 ω 是可微的. M 上的 p 次形式 ω 可以写作

$$\omega = \sum_{i_1<\cdots<i_p}\omega_{i_1,\cdots,i_p}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}.$$

这里函数 $\omega_{i_1,\cdots,i_p}$ 只定义在 M 上，以前所给的 $\mathrm{d}\omega$ 的定义在这里没有意义了，因为 $D_j(\omega_{i_1,\cdots,i_p})$ 没有意义. 但是还有一个合理的定义 $\mathrm{d}\omega$ 的方法.

定理 5-3 在 M 上有惟一的 $(p+1)$ 次形式 $\mathrm{d}\omega$ 使得对每个坐标系

1. 这段话的意思是说可微向量场有两种定义方式：一是在 $A\subset\mathbf{R}^n$ 上定义可微向量场 $F(x)$，如第 3 章所述，再要求当 $x\in M$ 时 $F(x)\in M_x$. 这时必有 $W\subset\mathbf{R}^k$（见定理 5-2）上惟一可微向量场 G 与之对应. 另一方式是，先在 M 上定义向量场（不一定可微）F，使 $F(x)\in M_x$. 再用相应的 W 上的向量场 G 的可微性作为 F 可微性的定义. ——译者注

$f: W \to \mathbf{R}^n$ 都有

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega).$$

证　若 $f: W \to \mathbf{R}^n$ 是一个坐标系使得 $x = f(a), v_1, \cdots, v_{p+1} \in M_x$，于是在 $\mathbf{R}^k{}_a$ 中有惟一的一组 $w_1, \cdots, w_{p+1}$ 使 $f_*(w_i) = v_i$. 我们定义 $\mathrm{d}\omega(x)(v_1, \cdots, v_{p+1}) = \mathrm{d}(f^*\omega)(a)(w_1, \cdots, w_{p+1})$. 可以验证 $\mathrm{d}\omega(x)$ 的这个定义并不依赖于坐标系 f，所以 $\mathrm{d}\omega$ 是完全定义了的. 此外，显然 $\mathrm{d}\omega$ 必须如此定义，所以 $\mathrm{d}\omega$ 又是惟一的. ▌

时常有必要在流形 M 的每个切空间 M_x 上选择一个定向 μ_x. 如果对于每个坐标系 $f: W \to \mathbf{R}^n$ 和 $a, b \in W$，

$$[f_*((e_1)_a), \cdots, f_*((e_k)_a)] = \mu_{f(a)}$$

当且仅当

$$[f_*((e_1)_b), \cdots, f_*((e_k)_b)] = \mu_{f(b)}$$

时成立，则这种选择方式就称为**协调**的（图 5-6）.

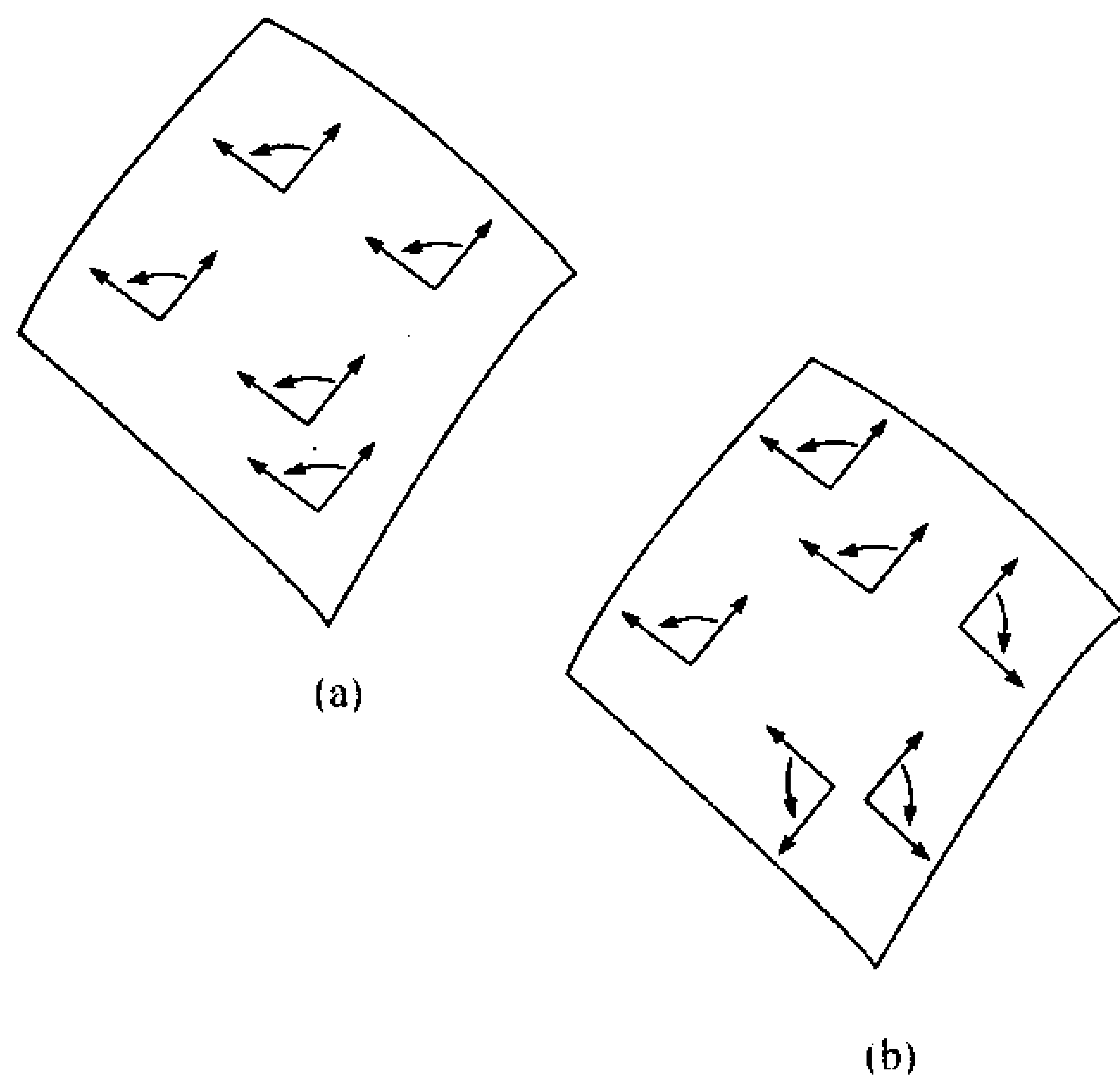


图 5-6　(a) 定向的协调选择 (b) 定向的不协调选择

设已协调地选定了定向 μ_x. 如果 $f: W \to \mathbf{R}^n$ 是这样一个坐标系，且 W 是连通的，使得只要对一个 $a \in W$，

$$[f_*((e_1)_a), \cdots, f_*((e_k)_a)] = \mu_{f(a)}$$

成立，则对于每个 $a \in W$ 上式也对，这时，坐标系 f 称为**保持定向**的．如果 f 不是保持定向的而 $T:\mathbf{R}^k \to \mathbf{R}^k$ 是一个线性变换，$\det T = -1$，则 $f \circ T$ 是保持定向的．所以在每一点周围总有一个保持定向的坐标系．如果 f 和 g 都是保持定向的，而且 $x = f(a) = g(b)$，则由关系式

$$[f_*((e_1)_a), \cdots, f_*((e_k)_a)] = \mu_x = [g_*((e_1)_b), \cdots, g_*((e_k)_b)],$$

必有

$$[(g^{-1} \circ f)_*((e_1)_a), \cdots, (g^{-1} \circ f)_*((e_k)_a)] = [(e_1)_b, \cdots, (e_k)_b],$$

所以 $\det(g^{-1} \circ f)' > 0$，这是一个要记住的重要事实．

可以协调地选定定向 μ_x 的流形称为**可定向的**，而协调地选定的这个 μ_x 就说是 M 的一个**定向** μ．流形连同定向 μ 就称为**有向**流形．默比乌斯（Möbius）带是不可定向流形的经典的例子．把一张纸条扭了半转的两端再粘起来就得到它的模型（图 5-7）．

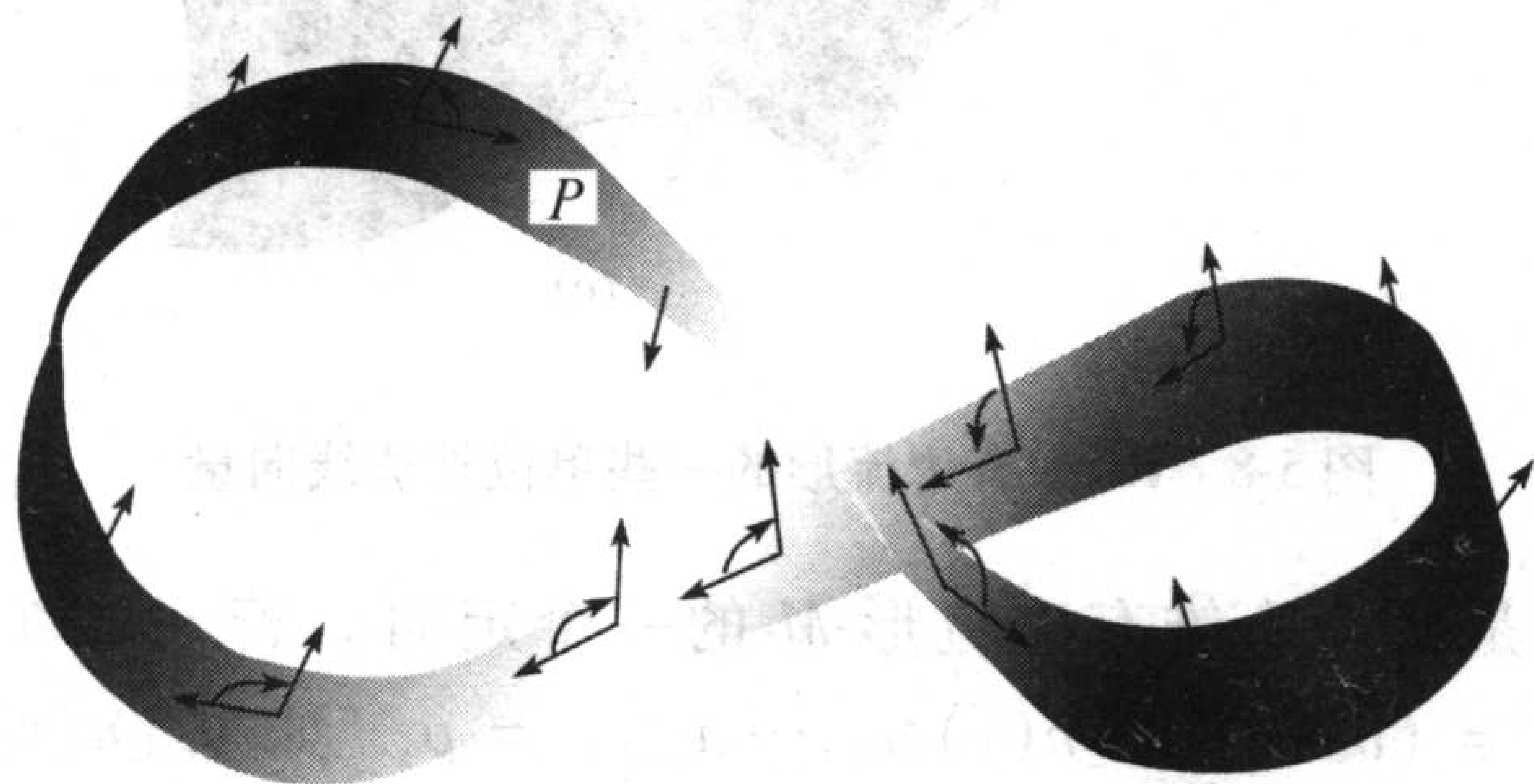


图 5-7　默比乌斯带是一个不可定向流形．一个基底从 P 开始向右运动，绕一圈再回到 P 点后，定向就反过来了

对有边流形也可以定义向量场、微分形式和定向．若 M 是一个 k 维有边流形而且 $x \in \partial M$，则 $(\partial M)_x$ 是 k 维向量空间 M_x 的 $(k-1)$ 维子空间．所以 M_x 恰好有两个单位向量垂直于 $(\partial M)_x$．它们可以这样区分（图 5-8）：若 $f: W \to \mathbf{R}^n$ 是一个坐标系，$W \subset H^k$，$f(0) = x$，则这两个单位向量中只有一个可写成 $f_*(v_0)$，v_0 是某个适合 $v^k < 0$ 的向量．

这一个单位向量称为**单位外法线向量**，记作 $n(x)$. 不难验证这个定义与坐标系 f 的选择关系.

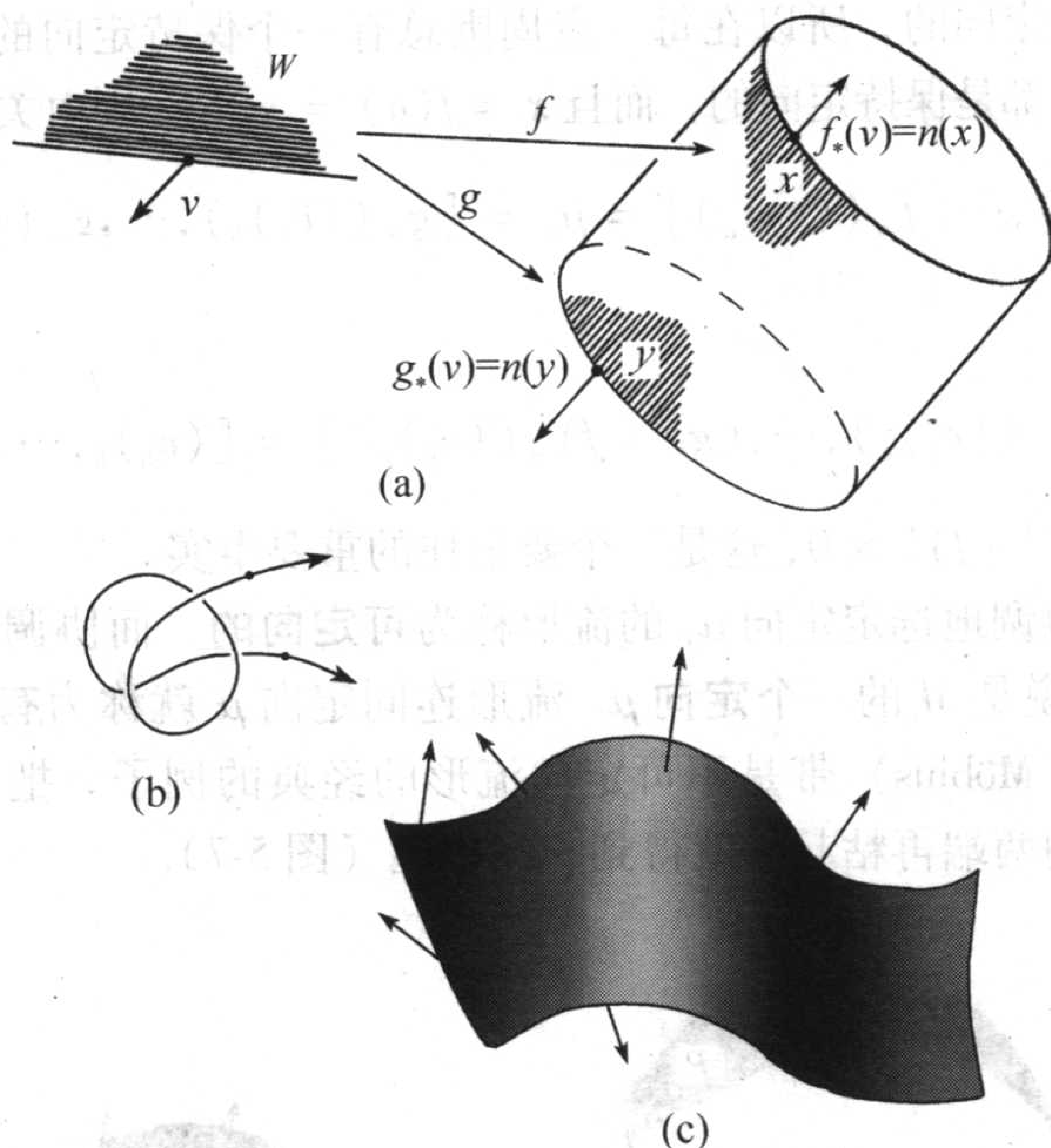


图 5-8　$\mathbf{R}^3$ 中有边流形的一些单位外法线向量

设 μ 是一个 k 维有边流形 M 的一个定向. 若 $x \in \partial M$, 选取 $v_1, \cdots, v_{k-1} \in (\partial M)_x$ 使 $[n(x), v_1, \cdots, v_{k-1}] = \mu_x$. 如果 $[n(x), w_1, \cdots, w_{k-1}] = \mu_x$ 也成立, 这里也有 $w_1, \cdots, w_{k-1} \in (\partial M)_x$, 则 $[v_1, \cdots, v_{k-1}]$ 和 $[w_1, \cdots, w_{k-1}]$ 是 $(\partial M)_x$ 的同一定向. 这个定向记作 $(\partial\mu)_x$. 容易看到, 对于 $x \in \partial M$, 这些定向 $(\partial\mu)_x$ 在 ∂M 上是协调的. 所以, 如果 M 可定向, ∂M 也可定向, 而 M 上一个定向 μ 决定了 ∂M 上一个定向 $\partial\mu$, 称为**诱导定向**. 如果把这些用于具有通常定向的 $\mathbf{H}^k$, 就知道 $\mathbf{R}^{k-1} = \{x \in \mathbf{H}^k : x^k = 0\}$ 上的诱导定向是 $(-1)^k$ 乘上 $\mathbf{R}^{k-1}$ 的通常定向. 选择这样一种定向的理由在下一节就会明白.

如果 M 是 $\mathbf{R}^n$ 中一个有向 $(n-1)$ 维流形, 即使它不必是某个 n

维流形的边缘，也可以定义单位外法线向量的代替物．如果 $[v_1,\cdots,v_{n-1}]=\mu_x$，我们在 $\mathbf{R}^n_x$ 中取一个垂直于 M_x 的单位向量 $n(x)$，并使 $[n(x),v_1,\cdots,v_{n-1}]$ 成为 $\mathbf{R}^n_x$ 的通常定向．我们仍把 $n(x)$ 称为(由 μ 决定的) M 的单位外法线向量．向量 $n(x)$ 在一种明显的意义下在 M 上连续变化．反之，如果在整个 M 上定义了一族连续变化的单位法向量 $n(x)$，那么我们也能决定 M 的一个定向．这说明，在默比乌斯带上，不可能选出这样一种连续变化的法向量．在默比乌斯带的纸带模型中，我们可以认为（有厚度）纸带两侧是方向相反的两个法向量的端点．纸带模型的一个著名性质为：纸带模型是单侧曲面（如果你从一侧开始在纸带上连续地涂色，最后一定会把两侧都涂满同样颜色），此性质反映了不可能连续选取法向量．换句话说，在一个点任意选定 $n(x)$ 后，由于在其他点上连续性的要求，最终将迫使在起点处选出相反的 $n(x)$．

习题

5-9. 证明 M_x 由 M 中的曲线 c 在 t 处的切向量组成，这里 $c(t)=x$.

5-10. 设 $\mathcal{C}$ 是 M 的坐标系的一个集合，它适合：(1) 对每一个 $x\in M$ 都有 x 周围的一个坐标系 $f\in\mathcal{C}$；(2) 若 $f,g\in\mathcal{C}$ 都是 $x\in M$ 周围的坐标系，则 $\det(f^{-1}\circ g)'>0$. 证明 M 上有一个定向，使得当 $f\in\mathcal{C}$ 时，f 为保持定向的．

5-11. 若 M 是 $\mathbf{R}^n$ 中的一个可定向的 n 维有边流形，定义 μ_x 为 $M_x=\mathbf{R}^n_x$ 的通常定向(这样定义的定向 μ 就是 M 的**通常定向**). 若 $x\in\partial M$, 求证上面给的 $n(x)$ 的两种定义是一致的．

5-12. (a) 若 F 是 $M\subset\mathbf{R}^n$ 上的可微向量场，证明必存在一个开集 $A\supset M$ 及 A 上的可微向量场 $\widetilde{\mathcal{F}}$，使得对于 $x\in M$，$\widetilde{\mathcal{F}}(x)=F(x)$. 提示：先局部地做，再用单位分解．

(b) 若 M 为闭的，证明可取 $A=\mathbf{R}^n$.

5-13. 令 $g:A\to\mathbf{R}^p$ 如定理 5-1.

(a) 若 $x\in M=g^{-1}(0)$，令 $h:U\to\mathbf{R}^n$ 为本质上惟一地使得 $g\circ h(y)=(y^{n-p+1},\cdots,y^n)$, $h(0)=x$ 的微分同胚．定义 $f:\mathbf{R}^{n-p}\to\mathbf{R}^n$ 为 $f(a)=h(0,a)$. 证明 f_* 是 1-1 的，且使 $n-p$ 个向量 $f_*((e_1)_0),\cdots,f_*((e_{n-p})_0)$

线性无关.

(b) 证明可以协调地定义定向μ_x，以致M是可定向的.

(c) 若$p=1$，证明在x处的单位外法线向量是$D_1g(x),\cdots,D_ng(x)$的某个倍数.

5-14. 若$M\subset\mathbf{R}^n$是一个可定向$(n-1)$维流形，证明必定存在一个开集$A\subset\mathbf{R}^n$以及一个可微的$g:A\to\mathbf{R}^1$使$M=g^{-1}(0)$，而且$g'(x)$的秩对于$x\in M$为1. 提示：习题5-4局部地证明了此结论. 用定向来选出协调的局部解答，再用单位分解.

5-15. 令M是$\mathbf{R}^n$中一个$(n-1)$维流形，$M(\varepsilon)$是两个相反方向上的长度为ε的所有法线向量端点的集合，再设ε充分小使$M(\varepsilon)$也是一个$(n-1)$维流形. 求证$M(\varepsilon)$是可定向的(即使M不可定向). 如果M是默比乌斯带，$M(\varepsilon)$是什么?

5-16. 令$g:A\to\mathbf{R}^p$定义如定理5-1. 如果$f:\mathbf{R}^n\to\mathbf{R}$是可微的，而$f$在$g^{-1}(0)$上的极大(极小)值在$a$点达到. 证明必定存在$\lambda_1,\cdots,\lambda_p\in\mathbf{R}$使得

$$(1)\ D_jf(a)=\sum_{i=1}^{n}\lambda_iD_jg^i(a)\qquad j=1,\cdots,n.$$

提示：这个方程可以写成$\mathrm{d}f(a)=\sum_{i=1}^{n}\lambda_i\mathrm{d}g^i(a)$，当$g(x)=(x^{n-p+1},\cdots,x^n)$时，它是显然的.

f在$g^{-1}(0)$上的极大（极小）值有时称为f在**约束**$g^i=0$下的极大（极小）值. 可以试从解方程(1)来求a. 特别是，如果$g:A\to\mathbf{R}$，我们必须从下面$n+1$个方程解出$n+1$个未知数$a^1,\cdots,a^n,\lambda$：

$$D_jf(a)=\lambda D_jg(a),\qquad j=1,\cdots,n,$$

$$g(a)=0.$$

如果把$g(a)=0$放在最后解，它时常很好解. 这就是**拉格朗日（Lagrange）方法**，那个有用的然而最终不出现的常数λ称为**拉格朗日算子**. 下一题是拉格朗日算子在理论上的一个很好的应用.

5-17. (a) 令$T:\mathbf{R}^n\to\mathbf{R}^n$是自伴的，其矩阵$A=(a_{ij})$，则$a_{ij}=a_{ji}$. 若$f(x)=\langle Tx,x\rangle=\sum a_{ij}x^ix^j$，证明$D_kf(x)=2\sum_{j=1}^{n}a_{kj}x^j$. 考虑$\langle Tx,x\rangle$在$S^{n-1}$上的极大值，证明存在$x\in S^{n-1}$和$\lambda\in\mathbf{R}$使$Tx=\lambda x$.

(b) 若$V=\{y\in\mathbf{R}^n:\langle x,y\rangle=0\}$，证明$T(V)\subset V$，而且$T:V\to V$是自伴的.

（c）证明 T 有一个由特征向量构成的基底.

5.3 流形上的斯托克斯定理

如果 ω 是 k 维有边流形 M 上的 p 次形式，c 是 M 中的一个奇异 p 维立方体，和前面一样我们定义

$$\int_c \omega = \int_{[0,1]^p} c^* \omega,$$

p 维链上的积分也和前面一样定义. 在 $p=k$ 的情况下，可能有一个开集 $W \supset [0,1]^k$ 和一个坐标系 $f: W \to \mathbf{R}^n$ 使得对于 $x \in [0,1]^k$ 有 $c(x) = f(x)$. M 中的 k 维立方体总指的是属于这种类型的. 如果 M 是有向的，当 f 保持定向时，就称奇异 k 维立方体 c 为**保持定向的**.

定理 5-4 *若 $c_1, c_2: [0,1]^k \to M$ 是两个保持定向的奇异 k 维立方体，这里 M 是有向 k 维流形，ω 是 M 上的一个 k 次形式，使得在 $c_1([0,1]^k) \cap c_2([0,1]^k)$ 外，$\omega=0$，则*

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

证 我们有

$$\int_{c_1} \omega = \int_{[0,1]^k} c_1^* \omega = \int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^* (\omega).$$

（这里 $c_2^{-1} \circ c_1$ 只定义在 $[0,1]^k$ 的一个子集上，而第二个等式用到了 $\omega = 0$ 于 $c_1([0,1]^k) \cap c_2([0,1]^k)$ 之外这个事实）. 因此只需证明

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^* (\omega) = \int_{[0,1]^k} c_2^* (\omega) = \int_{c_2} \omega.$$

若 $c_2^*(\omega) = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$ 而把 $c_2^{-1} \circ c_1$ 记为 g，则由定理4-9，又由于 $\det g' = \det(c_2^{-1} \circ c_1)' > 0$，我们有

$$\begin{aligned}(c_2^{-1}\circ c_1)^* c_2^*(\omega) &= g^*(f\,dx^1\wedge\cdots\wedge dx^k)\\ &= (f\circ g)\cdot\det g'\cdot dx^1\wedge\cdots\wedge dx^k\\ &= (f\circ g)\cdot|\det g'|\cdot dx^1\wedge\cdots\wedge dx^k,\end{aligned}$$

由定理 3-13 即得结论．▌

在证明中的最后一个等式应能说明为什么对待定向要这样小心．

令 ω 为有向 k 维流形 M 上的一个 k 次形式．如果有 M 中的一个保持定向的奇异 k 维立方体 c，使在 $c([0,1]^k)$ 之外 $\omega=0$，我们就定义

$$\int_M\omega=\int_c\omega.$$

定理5-4说明 $\int_M\omega$ 并不依赖于 c 的选取．现设 ω 是 M 上任一个 k 次形式．M 上必有一个开覆盖$\mathcal{O}$使对每一个 $U\in\mathcal{O}$ 都有一个**保持定向的奇异 k 维立方体** c 使 $U\cap M\subset c([0,1]^k)$．令 Φ 是 M 上从属于这个开覆盖的单位分解．定义

$$\int_M\omega=\sum_{\varphi\in\Phi}\int_M\varphi\cdot\omega,$$

等式右边要求如定理 3-12 前的讨论中所说的那样收敛(当 M 为紧时它当然收敛)．用类似于定理 3-12 的证法可以证明 $\int_M\omega$ 并不依赖于覆盖$\mathcal{O}$或 Φ.

所有这些定义都可以针对有定向 μ 的 k 维有边流形 M. 令∂M 有诱导定向$\partial\mu$. 令 c 是 M 中一个保持定向的奇异 k 维立方体，使得 $c_{(k,0)}$在∂M 中，而且是 c 的惟一的在∂M 中有内点的面．正如在$\partial\mu$ 的定义后的说明所指出的，当 k 为偶时 $c_{(k,0)}$是保持定向的，k 为奇时则不然．因此，若 ω 是 M 上一个$(k-1)$次形式而且在 $c([0,1]^k)$ 之外 $\omega=0$，则

$$\int_{c(k,0)}\omega=(-1)^k\int_{\partial M}\omega.$$

另一方面，$c_{(k,0)}$ 在∂c 中的系数是$(-1)^k$，所以

$$\int_{\partial c}\omega = \int_{(-1)^k c(k,0)}\omega = (-1)^k\int_{c(k,0)}\omega = \int_{\partial M}\omega.$$

我们对$\partial\mu$ 的选择就是为了使这个等式和下面的定理中不出现负号.

定理 5-5（斯托克斯定理） 若 M 是一个紧的有向的 k 维有边流形，ω 是 M 上一个（$k-1$）次形式，则

$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega.$$

[这里∂M要赋以诱导定向.]

证 先设 $M-\partial M$ 中有一个保持定向的奇异 k 维立方体使在 $c([0,1]^k)$ 之外 $\omega=0$. 由定理4-13 和 $\mathrm{d}\omega$ 的定义，我们有

$$\int_c \mathrm{d}\omega = \int_{[0,1]^k} c^*(\mathrm{d}\omega) = \int_{[0,1]^k}\mathrm{d}(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c}\omega.$$

因为在∂c 上 $\omega=0$，所以

$$\int_M \mathrm{d}\omega = \int_c \mathrm{d}\omega = \int_{\partial c}\omega = 0,$$

另一方面，因为在 ∂M 上 $\omega=0$，所以 $\int_{\partial M}\omega = 0$，而上式成为

$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega.$$

其次，设有一个保持定向的奇异 k 维立方体在 M 中，使得 $c_{(k,0)}$ 是在 ∂M 中的惟一的面，而且在 $c([0,1]^k)$ 之外 $\omega=0$. 则

$$\int_M \mathrm{d}\omega = \int_c \mathrm{d}\omega = \int_{\partial c}\omega = \int_{\partial M}\omega.$$

现在考虑一般情况. 必有 M 的一个开覆盖$\mathcal{O}$以及 M 上从属于$\mathcal{O}$的单位分解 Φ，使得对于每个$\varphi\in\Phi$，形式$\varphi\cdot\omega$ 是上述两种类型之一. 我们有

$$0 = \mathrm{d}(1) = \mathrm{d}\Big(\sum_{\varphi\in\Phi}\varphi\Big) = \sum_{\varphi\in\Phi}\mathrm{d}\varphi,$$

所以

$$\sum_{\varphi \in \Phi} \mathrm{d}\varphi \wedge \omega = 0.$$

因为 M 为紧的，这只是一个有限和，而我们有

$$\sum_{\varphi \in \Phi} \int_M \mathrm{d}\varphi \wedge \omega = 0.$$

所以

$$\begin{aligned}\int_M \mathrm{d}\omega &= \sum_{\varphi \in \Phi} \int_M \varphi \cdot \mathrm{d}\omega = \sum_{\varphi \in \Phi} \int_M \mathrm{d}\varphi \wedge \omega + \varphi \cdot \mathrm{d}\omega \\ &= \sum_{\varphi \in \Phi} \int_M \mathrm{d}(\varphi \cdot \omega) = \sum_{\varphi \in \Phi} \int_{\partial M} \varphi \cdot \omega = \int_{\partial M} \omega. \quad \blacksquare\end{aligned}$$

习题

5-18. 若 M 是 $\mathbf{R}^n$ 中一个 n 维流形（或有边流形），具有通常的定向，证明本节所定义的 $\int_M f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ 和第 3 章定义的 $\int_M f$ 一样.

5-19. (a) 证明：若 M 非紧，则定理 5-5 不真. 提示：若 M 是一个有边流形而定理 5-5 成立，则 $M - \partial M$ 也是一个有边流形（但具有空的边缘）.

(b) 证明若 ω 在 M 的一个紧子集外为 0，则当 M 非紧时定理 5-5 仍成立.

5-20. 若 ω 是一个紧 k 维流形 M 上的一个 $(k-1)$ 次形式，证明 $\int_M \mathrm{d}\omega = 0$. 若 M 非紧，作一反例.

5-21. V 上的**绝对 k 阶张量**就是一个形式 $|\omega|[\omega \in \Omega^k(V)]$ 的函数 $\eta: V^k \to \mathbf{R}^1$[1]，$M$ 上的**绝对 k 次形式**就是一个函数 η，$\eta(x)$ 是 M_x 上的绝对 k 阶张量. 证明即使 M 为不可定向，仍可定义 $\int_M \eta$.

5-22. 若 $M_1 \subset \mathbf{R}^n$ 是一个 n 维有边流形，而 $M_2 \subset M_1 - \partial M_1$ 也是一个 n 维有边流形，且 M_1, M_2 均为紧的，求证

$$\int_{\partial M_1} \omega = \int_{\partial M_2} \omega,$$

1. $|\omega|(v_1, \cdots, v_k) = |\omega(v_1, \cdots, v_k)|$，第 2 章习题 2-13 (d) 中对 $f: \mathbf{R} \to \mathbf{R}$ 引用了类似记号.——译者注

这里 ω 是 M_1 上的一个 $(n-1)$ 次形式，而 ∂M_1, ∂M_2 的定向是 M_1 与 M_2 的通常定向所诱导的. 提示：找一个有边流形 M 使 $\partial M = \partial M_1 \cup \partial M_2$，且使 ∂M 的诱导定向在 ∂M_1 上与 ∂M_1 上与原有的诱导定向一致，在 ∂M_2 上的则与 ∂M_2 上诱导定向相反.

5.4 体积元素

令 M 是 $\mathbf{R}^n$ 中的一个 k 维流形（或有边流形），其定向为 μ. 若 $x \in M$，则 μ_x 和前面所定义的内积 T_x 决定了一个体积元素 $\omega(x) \in \Omega^k(M_x)$. 于是我们得到一个在 M 上处处不为 0 的 k 次形式 ω，称为 M 上的（由 μ 决定的）**体积元素**，记作 dV，纵然它一般不是一个 $(k-1)$ 次形式的微分. M 的**体积**定义为 $\int_M dV$，只要积分存在. 当 M 为紧时，它确实是存在的. 对于一维和二维流形，"体积" 通常叫做**曲线弧长**和**曲面面积**，dV 则记作 ds（"弧长元素"）和 dA [或 dS]（"[曲面]面积元素"）.

一个有趣的具体情况是 $\mathbf{R}^3$ 中有向曲面（二维流形）M 的体积元素值得关注. 令 $n(x)$ 为点 $x \in M$ 处的单位外法线向量. 若 $\omega \in \Omega^2(M_x)$ 定义为

$$\omega(v,w) = \det\begin{pmatrix} v \\ w \\ n(x) \end{pmatrix},$$

则当 v 与 w 是 M_x 的标准正交基底而且 $[v,w]=\mu_x$ 时 $\omega(v,w)=1$. 于是 $dA=\omega$. 另一方面，由 $v\times w$ 之定义，$\omega(v,w) = \langle v\times w,\ n(x)\rangle$. 于是我们有

$$dA(v,w) = \langle v \times w, n(x)\rangle .$$

因为当 $v, w \in M_x$ 时 $v\times w$ 是 $n(x)$ 的倍数，我们得知

$$dA(v,w) = |v \times w|,$$

只要 $[v,w]=\mu_x$. 如果我们想计算 M 的面积，就必须对保持定向的奇异2维立方体 c 计算

$$\int_{[0,1]^2} c^*(\mathrm{d}A).$$

我们定义

$$E(a)=[D_1c^1(a)]^2+[D_1c^2(a)]^2+[D_1c^3(a)]^2,$$
$$F(a)=D_1c^1(a)\cdot D_2c^1(a)+D_1c^2(a)\cdot D_2c^2(a)+D_1c^3(a)\cdot D_2c^3(a),$$
$$G(a)=[D_2c^1(a)]^2+[D_2c^2(a)]^2+[D_2c^3(a)]^2.$$

于是，由习题4-9，有

$$\begin{aligned} c^*(\mathrm{d}A)((e_1)_a,(e_2)_a) &= \mathrm{d}A(c_*((e_1)_a),c_*((e_2)_a)) \\ &= |(D_1c^1(a),D_1c^2(a),D_1c^3(a)) \\ &\quad \times(D_2c^1(a),D_2c^2(a),D_2c^3(a))| \\ &= \sqrt{E(a)G(a)-F(a)^2}. \end{aligned}$$

所以

$$\int_{[0,1]^2} c^*(\mathrm{d}A)=\int_{[0,1]^2}\sqrt{EG-F^2}.$$

计算曲面面积显然是麻烦极了的事，所幸极少需要知道曲面的面积. 此外，$\mathrm{d}A$ 还有一个简单的表达式，足供理论探讨之用.

定理 5-6　令 M 为 $\mathbf{R}^3$ 中的一个有向二维流形（或有边流形），令 n 表示单位外法线向量. 则

(1) $\mathrm{d}A=n^1\mathrm{d}y\wedge\mathrm{d}z+n^2\mathrm{d}z\wedge\mathrm{d}x+n^3\mathrm{d}x\wedge\mathrm{d}y.$

此外，在 M 上我们有

(2) $n^1\mathrm{d}A=\mathrm{d}y\wedge\mathrm{d}z.$

(3) $n^2\mathrm{d}A=\mathrm{d}z\wedge\mathrm{d}x.$

(4) $n^3\mathrm{d}A=\mathrm{d}x\wedge\mathrm{d}y.$

证　等式（1）等价于等式

$$dA(v,w) = \det\begin{pmatrix} v \\ w \\ n(x) \end{pmatrix}.$$

只要把行列式按最下一行展开就可证明这点．为证其他各式，令 $z \in \mathbf{R}^3_x$. 因为 $v \times w = \alpha n(x)$，$\alpha \in \mathbf{R}$，有

$$\begin{aligned}\langle z,n(x)\rangle \cdot \langle v \times w,n(x)\rangle &= \langle z,n(x)\rangle\alpha \\ &= \langle z,\alpha n(x)\rangle = \langle z,v \times w\rangle.\end{aligned}$$

分别取 $z = e_1, e_2, e_3$ 即得 (2), (3), (4)式．▌

有一点要当心：若 $\omega \in \Omega^2(\mathbf{R}^3_a)$ 定义为

$$\begin{aligned}\omega = n^1(a) \cdot dy(a) \wedge dz(a) + n^2(a) \cdot dz(a) \wedge dx(a) \\ + n^3(a) \cdot dx(a) \wedge dy(a),\end{aligned}$$

就不真，例如：

$$n^1(a) \cdot \omega = dy(a) \wedge dz(a).$$

上式双方只有在作用到 $v, w \in M_a$ 时才能得到相同的结果．

还应该用几句话来说明我们所给出的曲线弧长和曲面面积定义的合理性．若 $c:[0,1] \to \mathbf{R}^n$ 是可微的，而且 $c([0,1])$ 是一维有边流形，可以证明，但很繁琐，$c([0,1])$ 的弧长正是内接于它的折线长度的上确界．若 $c:[0,1]^2 \to \mathbf{R}^n$，人们自然希望，若作许多三角形使其顶点都在曲面 $c([0,1]^2)$ 上，则由这类三角形所组成的面，其面积的上确界就是曲面 $c([0,1]^2)$ 的面积．使人颇为惊奇的是，这种上确界通常不存在——可以找到这样的内接多面体，使它任意接近曲面 $c([0,1]^2)$，但其表面积可任意大！图 5-9 中的柱面就表明这一点．先后有多种曲面面积的定义，彼此不相一致，但是对于可微曲面而言都和我们的定义一致．关于这些难点问题的讨论，读者可以参看文献[3]或[15].

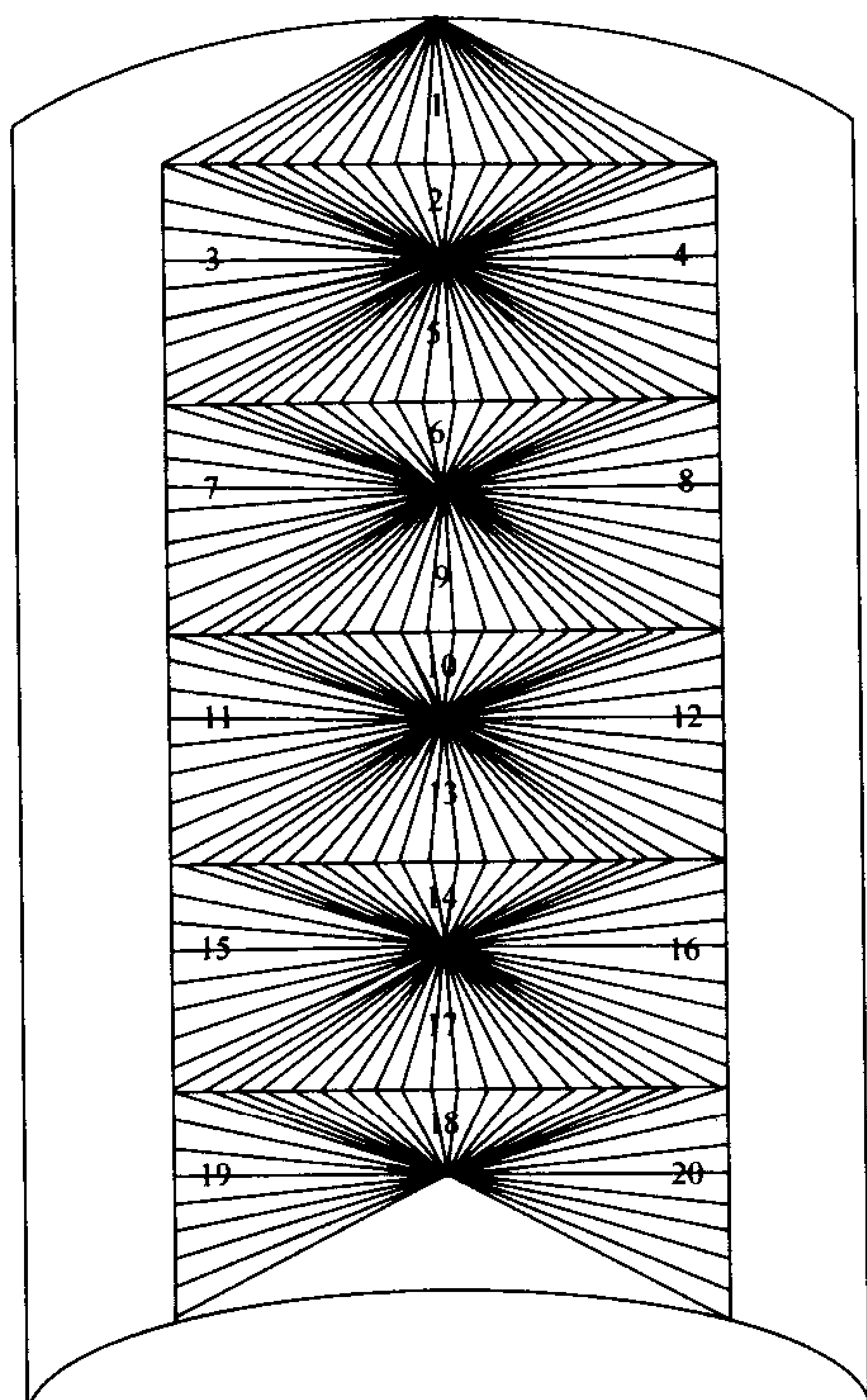


图5-9　在柱面上一部分区域中内接了20个三角形的曲面．令三角形个数足够多，并使三角形3，4，7，8等的底足够小，可以使内接面的面积任意大

习题

5-23. 若 M 是 $\mathbf{R}^n$ 中的一个有向一维流形，$c:[0,1]^2 \to M$ 是保持定向的，求证

$$\int_{[0,1]} c^*(\mathrm{d}s) = \int_{[0,1]} \sqrt{[(c^1)']^2 + \cdots + [(c^n)']^2}.$$

5-24. 若 M 是 $\mathbf{R}^n$ 中具有通常定向的 n 维流形，证明 $\mathrm{d}V = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$，所以本节所定义的 M 的体积就是第3章定义的体积．（注意这个结果依赖于 $\omega \wedge \eta$ 定义中的数字因子．）

5-25. 把定理5-6推广到 $\mathbf{R}^n$ 中的有向 $(n-1)$ 维流形．

5-26. (a) 若 $f:[a,b] \to \mathbf{R}$ 非负而且 f 在 xy 平面上的图像绕 x 轴在 $\mathbf{R}^3$ 中旋转产生曲面 M，求证 M 的面积是

$$\int_a^b 2\pi f\sqrt{1+(f')^2}.$$

(b) 计算 S^2 的面积.

5-27. 若 $T:\mathbf{R}^n \to \mathbf{R}^n$ 是保持范数不变的线性变换，而 M 是 $\mathbf{R}^n$ 中的 k 维流形，证明 M 和 $T(M)$ 体积相同.

5-28. (a) 若 M 是一个 k 维流形，证明可以定义一个 k 阶绝对张量 $|\mathrm{d}V|$，即使 M 不是可定向的也行，从而 M 的体积可定义为

$$\int_M |\mathrm{d}V|.$$

(b) 若 $c:[0,2\pi]\times(-1,1)\to\mathbf{R}^3$ 定义为

$c(u,v) = (2\cos u + v\sin(u/2)\cos u, 2\sin u + v\sin(u/2)\sin u, v\cos u/2)$,

证明 $c[0,2\pi]\times(-1,1)$ 是默比乌斯带，并求其面积.

5-29. 若在 k 维流形 M 上有一个处处不为 0 的 k 次形式，证明 M 是可定向的.

5-30. (a) 若 $f:[0,1]\to\mathbf{R}$ 可微而 $c:[0,1]\to\mathbf{R}^2$ 定义为 $c(x) = (x, f(x))$，证明 $c([0,1])$ 的弧长是 $\int_0^1\sqrt{1+(f')^2}$.

(b) 证明这个弧长是其内接折线长的上确界. 提示：若 $0=t_0\leqslant t_1\leqslant\cdots\leqslant t_n=1$，则有 $s_i\in[t_{i-1},t_i]$ 使

$$\begin{aligned}|c(t_i)-c(t_{i-1})| &= \sqrt{(t_i-t_{i-1})^2+(f(t_i)-f(t_{i-1}))^2}\\ &= \sqrt{(t_i-t_{i-1})^2+f'(s_i)^2(t_i-t_{i-1})^2}.\end{aligned}$$

5-31. 考虑 $\mathbf{R}^3-0$ 中的 2 次形式 ω 定义为：

$$\omega = \frac{x\,\mathrm{d}y\wedge\mathrm{d}z + y\,\mathrm{d}z\wedge\mathrm{d}x + z\,\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{3/2}}.$$

(a) 求证 ω 是闭形式.

(b) 求证

$$\omega(p)(v_p,w_p) = \frac{\langle v\times w,p\rangle}{|p|^3}.$$

对于 $r>0$，记 $S^2(r) = \{x\in\mathbf{R}^3: |x| = r\}$. 证明 ω 限制在 $S^2(r)$ 的切空间上就成为 $1/r^2$ 乘以体积元素，而且 $\int_{S^2(r)}\omega = 4\pi$. 由此断定：$\omega$ 不是恰当的. 然而我们仍用 $\mathrm{d}\theta$ 记 ω，因为我们将看到，$\mathrm{d}\theta$ 是 $\mathbf{R}^2-0$ 中的一次形式 $\mathrm{d}\theta$ 的

类似物.

(c) 若 v_p 是切向量，对某个 $\lambda \in \mathbf{R}$ 使得 $v = \lambda p$，证明对于一切切向量 w_p 有 $\mathrm{d}\theta(p)(v_p, w_p) = 0$. 若 $\mathbf{R}^3$ 中二维流形 M 是广义锥的一部分，即由过原点的线段之并组成，证明 $\int_M \mathrm{d}\theta = 0$.

(d) 设 $M \subset \mathbf{R}^3 - 0$ 是一紧二维有边流形，使得每条过 0 的射线都与 M 至多只交一次(图 5-10). 这些过 0 而且交 M 的射线组成一立体锥 $C(M)$：M 所张的**立体角**定义为 $C(M) \cap S^2$ 的面积，与此等价，也就是 $1/r^2$ 乘以 $C(M) \cap S^2(r)$ 的面积，$r>0$. 证明 M 所张的立体角等于 $\left|\int_M \mathrm{d}\theta\right|$. 提示：取 r 充分小，以致有一个 3 维有边流形 N（见图5-10）使其边缘 ∂N 是 M 和 $C(M) \cap S^2(r)$ 以及一广义锥的一部分之并.（其实 N 是一个有角流形，见下一节末尾的说明.）

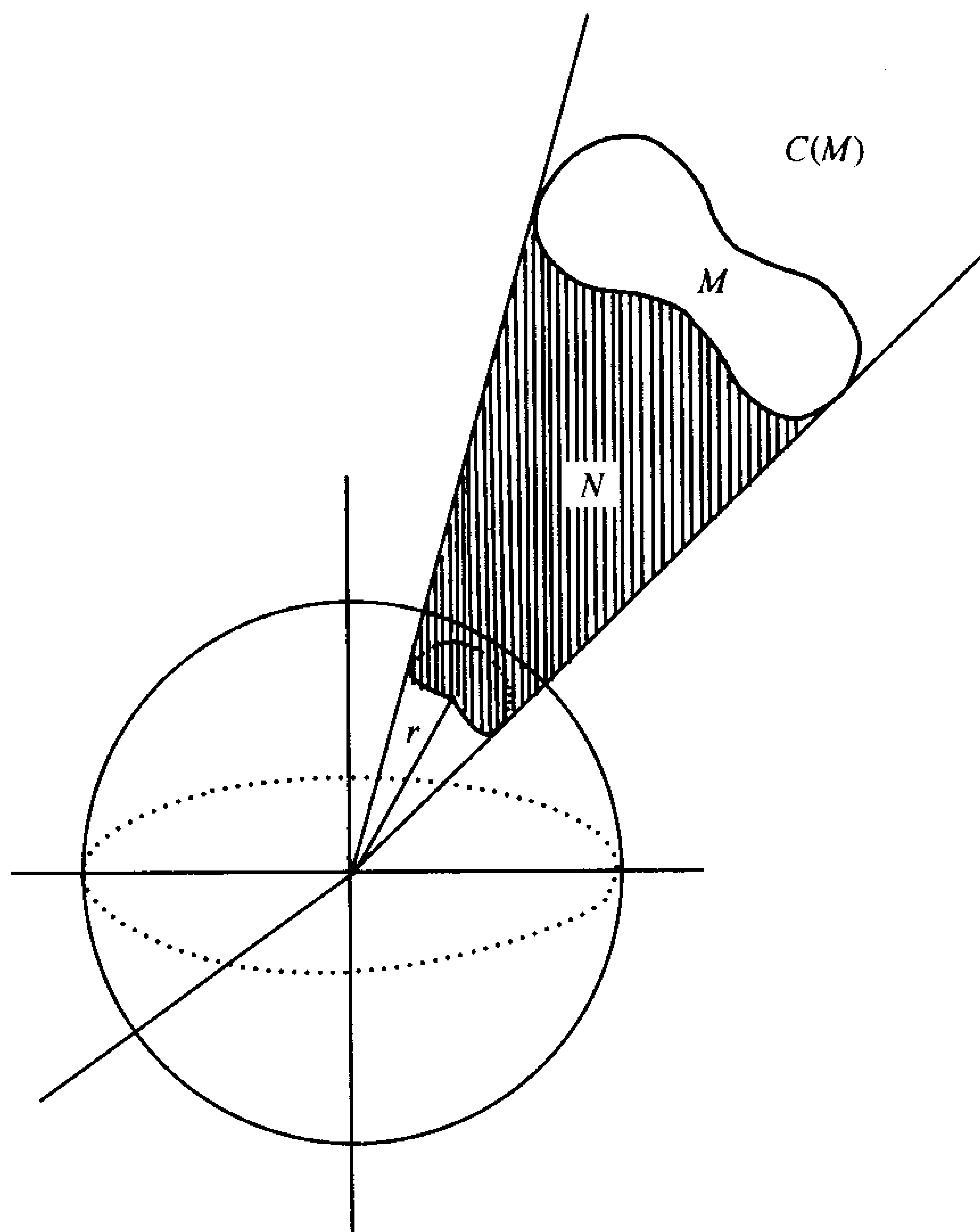


图 5-10

5-32. 令$f, g:[0,1] \to \mathbf{R}^3$是不相交封闭曲线．定义其连结数$l(f,g)$为（参看习题4-34）

$$l(f,g) = \frac{-1}{4\pi}\int_{c_{f,g}} d\theta.$$

(a) 若(F,G)是不相交封闭曲线的同伦，则$l(F_0,G_0) = l(F_1,G_1)$.

(b) 若$r(u,v) = |f(u) - g(v)|$，证明

$$l(f,g) = \frac{-1}{4\pi}\int_0^1\int_0^1 \frac{1}{[r(u,v)]^3} \cdot A(u,v)\,du\,dv$$

其中

$$A(u,v) = \det\begin{pmatrix} (f^1)'(u) & (f^2)'(u) & (f^3)'(u) \\ (g^1)'(v) & (g^2)'(v) & (g^3)'(v) \\ f^1(u) - g^1(v) & f^2(u) - g^2(v) & f^3(u) - g^3(v) \end{pmatrix}$$

(c) 证明若f与g都在xy平面上，则$l(f,g) = 0$. 图4-5(b)的两条曲线是$f(u) = (\cos u, \sin u, 0)$和$g(v) = (1 + \cos v, 0, \sin v)$. 很容易看到用上面那个积分来计算$l(f,g)$是没有希望的．下面习题说明如何不用明显的计算即可求出$l(f,g)$.

5-33. (a) 若$(a,b,c) \in \mathbf{R}^3$，定义

$$d\theta_{(a,b,c)} = \frac{(x-a)dy \wedge dz + (y-b)dz \wedge dx + (z-c)dx \wedge dy}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{3/2}}.$$

若M是$\mathbf{R}^3$中的紧的二维有边流形，$(a,b,c) \notin M$，定义

$$\Omega(a,b,c) = \int_M d\theta_{(a,b,c)}.$$

令(a,b,c)是在M的外法线一侧的一点，而(a',b',c')是在其另一侧上的一点．取(a,b,c)和(a',b',c')充分接近即可证明，可以使$\Omega(a,b,c) - \Omega(a',b',c')$任意接近于$-4\pi$. 提示：先证，若$M = \partial N$，则当$(a,b,c) \in N - M$时$\Omega(a,b,c) = -4\pi$，当$(a,b,c) \notin N$时$\Omega(a,b,c) = 0$.

(b) 设$f([0,1]) = \partial M$，M是一个紧有向二维有边流形．（如果f是不自交的曲线，这样的M总是存在的，即使f是打了结的，参看文献［6］第138页．）设当g与M在x点相交时，g在x点的切向量v不在M_x中．令n^+表示使上述v与M的外法向指向同侧的g与M的交点个数，n^-是其余

交点个数. 若 $n=n^{+}-n^{-}$，证明

$$n=\frac{-1}{4\pi}\int_{g}\mathrm{d}\Omega.$$

(c) 令 $r(x,y,z)=|(x,y,z)|$，求证

$$\mathrm{D}_1\Omega(a,b,c)=\int_f\frac{(y-b)\mathrm{d}z-(z-c)\mathrm{d}y}{r^3}$$

$$\mathrm{D}_2\Omega(a,b,c)=\int_f\frac{(z-c)\mathrm{d}x-(x-a)\mathrm{d}z}{r^3}$$

$$\mathrm{D}_3\Omega(a,b,c)=\int_f\frac{(x-a)\mathrm{d}y-(y-b)\mathrm{d}x}{r^3}$$

(d) 证明(b)中的整数 n 等于习题5-32(b)中的积分，并用这结果证明当f，g 为图4-6(b)中的曲线时，$l(f,g)=1$，当f，g 为图4-6(c)中的曲线时，$l(f,g)=0$.（这些结果由高斯（Gauss）[7]所知. 这里勾画的证法见文献［4］的第409—411页；又见［13］第2卷，41—43页.）

5.5 一些经典定理

现在我们已经完全做好准备来叙述和证明几个经典的“斯托克斯型”的定理. 我们将比较随便地采用一些无需解释的经典的记号.

定理5-7（格林定理） 令 $M\subset\mathbf{R}^2$ 是一个紧的二维有边流形. 假设 $\alpha,\beta:M\to\mathbf{R}$ 是可微的，则

$$\int_{\partial M}\alpha\,\mathrm{d}x+\beta\,\mathrm{d}y=\int_M(\mathrm{D}_1\beta-\mathrm{D}_2\alpha)\mathrm{d}x\wedge\mathrm{d}y=\iint\limits_M\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

（这里 M 赋以通常定向，∂M 赋以诱导定向，也就是逆时针定向.）

证 因为 $\mathrm{d}(\alpha\,\mathrm{d}x+\beta\,\mathrm{d}y)=(\mathrm{D}_1\beta-\mathrm{D}_2\alpha)\mathrm{d}x\wedge\mathrm{d}y$，所以这就是定理5-5的很特殊的情况. ▌

定理5-8（散度定理） 令 $M\subset\mathbf{R}^3$ 是一个紧的三维有边流形，n 是 ∂M 上的单位外法线向量. 设 F 是 M 上的可微向量场，则

$$\int_M\operatorname{div}F\,\mathrm{d}V=\int_{\partial M}\langle F,n\rangle\,\mathrm{d}A.$$

这个式子也可以用三个可微函数 $\alpha,\beta,\gamma:M\to\mathbf{R}$ 来写成

$$\iiint\limits_M\left(\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)\mathrm{d}V=\iint\limits_{\partial M}(n^1\alpha+n^2\beta+n^3\gamma)\,\mathrm{d}S.$$

证 在 M 上定义 ω 为 $\omega=F^1\mathrm{d}y\wedge\mathrm{d}z+F^2\mathrm{d}z\wedge\mathrm{d}x+F^3\mathrm{d}x\wedge\mathrm{d}y$，则 $\mathrm{d}\omega=\mathrm{div}\ F\mathrm{d}V$. 根据定理 5-6，在 ∂M 上我们有

$$\begin{aligned}n^1\mathrm{d}A&=\mathrm{d}y\wedge\mathrm{d}z,\\ n^2\mathrm{d}A&=\mathrm{d}z\wedge\mathrm{d}x,\\ n^3\mathrm{d}A&=\mathrm{d}x\wedge\mathrm{d}y.\end{aligned}$$

所以在 ∂M 上我们有

$$\begin{aligned}\langle F,n\rangle\mathrm{d}A&=F^1n^1\mathrm{d}A+F^2n^2\mathrm{d}A+F^3n^3\mathrm{d}A\\&=F^1\mathrm{d}y\wedge\mathrm{d}z+F^2\mathrm{d}z\wedge\mathrm{d}x+F^3\mathrm{d}x\wedge\mathrm{d}y\\&=\omega.\end{aligned}$$

因此，由定理 5-5，我们有

$$\int_M\mathrm{div}\ F\mathrm{d}V=\int_M\mathrm{d}\omega=\int_{\partial M}\omega=\int_{\partial M}\langle F,n\rangle\mathrm{d}A.\ \blacksquare$$

定理 5-9（斯托克斯定理） 令 $M\subset\mathbf{R}^3$ 是一个紧有向二维有边流形，n 是 M 上的由 M 之定向所决定的单位外法线．设 ∂M 具有诱导定向．令 T 为 ∂M 上的向量场，而且 $\mathrm{d}s(T)=1$，F 是定义在一个包含 M 的开集上的可微向量场．则

$$\int_M\langle(\nabla\times F),n\rangle\mathrm{d}A=\int_{\partial M}\langle F,T\rangle\mathrm{d}s.$$

这个式子有时写作

$$\begin{aligned}&\int_{\partial M}\alpha\mathrm{d}x+\beta\mathrm{d}y+\gamma\mathrm{d}z\\&=\iint\limits_M\left[n^1\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\right)+n^2\left(\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x}\right)+n^3\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)\right]\mathrm{d}S.\end{aligned}$$

证 在 M 上定义 ω 为 $\omega=F^1\mathrm{d}x+F^2\mathrm{d}y+F^3\mathrm{d}z$. 因为 $\nabla\times$ F 的分量

是$D_2F^3-D_3F^2, D_3F^1-D_1F^3, D_1F^2-D_2F^1$，如同在定理5-8的证明中一样，在M上我们有

$$\begin{aligned}\langle(\nabla\times F),n\rangle dA &= (D_2F^3-D_3F^2)dy\wedge dz\\ &+(D_3F^1-D_1F^3)dz\wedge dx+(D_1F^2-D_2F^1)dx\wedge dy=d\omega.\end{aligned}$$

另一方面，由于$ds(T)=1$，在∂M上我们有

$$T^1ds=dx, T^2ds=dy, T^3ds=dz.$$

(这些等式可以在$x\in\partial M$点上将它的双方作用到T_x上来验证，因为T_x是$(\partial M)_x$的基底．)
所以在∂M上我们有

$$\begin{aligned}\langle F,T\rangle ds &= F^1T^1ds+F^2T^2ds+F^3T^3ds\\ &= F^1dx+F^2dy+F^3dz=\omega.\end{aligned}$$

于是，由定理5-5，我们有

$$\int_M\langle(\nabla\times F),n\rangle dA=\int_M d\omega=\int_{\partial M}\omega=\int_{\partial M}\langle F,T\rangle ds.\ ▌$$

定理5-8和5-9是div F和curl F名称的由来．若$F(x)$是流体在x点(在某一时刻)的速度向量，则$\int_{\partial M}\langle F,n\rangle dA$就是从$M$“散出”的流体之量，从而条件div $F=0$表示流体不可压缩．若M是一个盘子，则$\int_{\partial M}\langle F,T\rangle ds$可用来度量绕盘心旋转的流体的量，如果对所有的盘它都为0，则$\nabla\times F=0$，流体也就称为无旋的．

对div F和curl F的这些解释归源于麦克斯韦（Maxwell）[13]．事实上麦克斯韦讨论的是div F的负值．因此他称之为敛度．对于$\nabla\times F$,麦克斯韦“十分犹豫地”建议用F之旋度的名词，由这个不幸的名词产生了一个缩写rot F，现在偶然还可以看到．

本节的几个经典定理时常可以陈述得比这里更普遍．例如格林定理对于正方形、散度定理对于立方体都成立．这两个事实可以用有边流形逼近正方形或立方体的办法来证明．本节定理的彻底的推

广需要有角流形的概念，它们是 $\mathbf{R}^n$ 的子集，而且局部地和 $\mathbf{R}^k$ 的一部分微分同胚，这一部分则是由若干片$(k-1)$维平面包围起来的．有志的读者可以自己去严格地定义有角流形，并且研究如何把这一整章结果推广，则会发现这是一个发人深省的练习．

习题

5-34 把散度定理推广到 $\mathbf{R}^n$ 中的 n 维有边流形．

5-35 将推广的散度定理应用于集 $M=\{x\in\mathbf{R}^n:|x|\leqslant a\}$ 以及 $F(x)=x_x$，用 $B_n=\{x\in\mathbf{R}^n:|x|\leqslant 1\}$ 的 n 维体积求出 $S^{n-1}=\{x\in\mathbf{R}^n:|x|=1\}$ 的体积．（B_n 的体积当 n 为偶数时等于 $\pi^{n/2}/(n/2)!$，当 n 为奇数时等于 $2^{(n+1)/2}\pi^{(n-1)/2}/1\cdot 3\cdot 5\cdot\cdots\cdot n$)．

5-36 在 $\mathbf{R}^3$ 上定义 F 为 $F(x)=(0,0,cx^3)_x$，M 是紧的三维有边流形，$M\subset\{x:x^3\leqslant 0\}$．向量场 F 可以看成是 $\{x:x^3\leqslant 0\}$ 中的密度为 c 的流体的向下的压力．[1] 因为流体在各个方向上都有相同压强，我们定义，流体加在 M 上的浮力为 $-\int_{\partial M}\langle F,n\rangle\,\mathrm{d}A$. 证明以下的阿基米德（Archimedes）定理：作用在 M 上的浮力等于 M 所排除的流体之重量．

1. 这里 F 是向量，就这点而言它表示力．但就其数值而言，它表示压强——即单位面积上的压力值．又这里假设重力加速度 $g=1$，在适当选取单位制时总可以做到这一点．要换到常用的单位制时不会有任何困难．——译者注

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索　　引

二　画

三　画

四　画

五　画

六　画

七 画

八 画

九 画

补　　遗

1. 在定理2-11(反函数定理)后应该提醒，f^{-1}的公式使我们能断定，f^{-1}实际上是连续可微的(而且若f是C^{∞}的，则f^{-1}也是的). 事实上，只须注意到一个矩阵A的逆之元素是原矩阵A之元素的C^{∞}函数即可，这一点从下面的“克莱姆法则”推出：$(A^{-1})_{ji} = (\det A^{ij}) / (\det A)$，$A^{ij}$是从$A$划掉第$i$行和第$j$列所得的矩阵.

2. 定理3-8前一部分的证明可以大为简化，而使引理3-7成为不必要的了. 只要用这样的闭矩形U_i的内域来覆盖B就可以了，这些闭矩形适合$\sum_{i=1}^{\infty} v(U_i) < \varepsilon$，再对每一点$x \in A - B$选一个以$x$为内点的闭矩形$V_x$使得$M_{V_x}(f) - m_{V_x}(f) < \varepsilon$. 如果分法$P$的每个子矩形都包含在$U_i$和$V_x$之某个有限集的一个之中，而这有限集又覆盖$A$，且对$A$中一切$x$有$|f(x)| \leqslant M$，则$U(f,P) - L(f,P) < \varepsilon v(A) + 2M\varepsilon$.

逆定理证明部分有一个错，因为只有当S之内域与$B_{1/n}$相交时才能保证$M_S(f) - m_S(f) \geqslant 1/n$. 为了弥补这一点，只须把$P$的所有子矩形的边界用总体积$< \varepsilon$的有限个矩形覆盖起来即可. 这些矩形连同δ将$B_{1/n}$覆盖而且总体积$< 2\varepsilon$.

3. 定理3-14(萨德定理)第一部分的论证需要详细一点. 若$U \subset A$是一个边长为l的闭矩形，则因为U是紧的，所以有一个具有以下性质的正整数N存在：若将U分为N^n个边长为l/N的矩形，则只要w和z在同一个这样的矩形S中，必有$|D_j g^i(w) - D_j g^i(z)| < \varepsilon/n^2$，已给$x \in S$，令$f(z) = Dg(x)(z) - g(z)$，于是当$z \in S$时有

$$|D_j f^i(z)| = |D_j g^i(x) - D_j g^i(z)| < \varepsilon/n^2.$$

故由引理2-10，若$x, y \in \delta$，则有

$$|Dg(x)(y-x) - g(y) + g(x)|$$

$$= |f(y) - f(x)| < \varepsilon |x - y| \leqslant \varepsilon \sqrt{n}(l/N).$$

4. 最后，本书中出现的记号 $\Lambda^k(V)$ 是不正确的，因为它与 $\Lambda^k(V)$ 的标准的定义（作为 V 之张量代数的某个商）不符. 我们讲的这个向量空间（对于有限维向量空间 V，它自然同构于 $\Lambda^k(V^*)$），记号 $\Omega^k(V)$ 可能日益成为标准的记号了.[1]

1. 本书已作替换. —— 编者注

附录　部分习题的解答或提示

——译者

1. 欧几里得空间上的函数

1-2 应为 $x = \lambda y$ 或 $y = \mu x$, 而 $\lambda, \mu \geqslant 0$.

1-5 因为 $z - x = (z - y) + (y - x)$, 所以

$$|z - x| = |(z - y) + (y - x)|.$$

再利用 1-1 定理的(3)即得.

1-6 (a)(1) 设对某个 $\lambda \in \mathbf{R}$ 有 $\int_a^b (f - \lambda g)^2 = 0$. 由积分的性质只能得到 $f - \lambda g = 0$ 几乎处处成立. 但若 f 与 g 均为连续, 则 $(f - \lambda g)^2$ 也连续而且非负. 一个非负连续函数当且仅当它恒为0时积分才为0. 所以 $f = \lambda g$ 处处成立. 这时自然有 $\left|\int_a^b f \cdot g\right| = |\lambda| \int_a^b g^2$, $(\int_a^b f^2)^{\frac{1}{2}} = (\lambda^2 \int_a^b g^2)^{\frac{1}{2}} = |\lambda| (\int_a^b g^2)^{\frac{1}{2}}$. 所以得到原题中的结论.

(2) 若对一切 $\lambda \in \mathrm{R}$, $\int_a^b (f - \lambda g)^2 > 0$, 即

$$\int_a^b f^2 - \lambda \int_a^b f \cdot g + \lambda^2 \int_a^b g^2 > 0.$$

视它为 λ 的二次三项式,则此二次三项式恒正,因此其判别式为负,所以又得到原题中的结论.

1-7 (a)因为 T 是保范数的,所以对一切 x, y 有 $|T(x + y)| = |x + y|$. 但是

$$|T(x + y)|^2 = |Tx|^2 + 2 < Tx, Ty > + |Ty|^2,$$
$$|x + y|^2 = |x|^2 + 2 < x, y > + |y|^2.$$

比较二式,注意到 T 是保范数的,所以 $|Tx|=|x|$, $|Ty|=|y|$, 所以有

$$<Tx,Ty> = <x,y>$$

对于一切 x,y 均成立,所以 T 也是保内积的.

反过来,设 T 是保内积的,即对任意 x,y 有

$$<Tx,Ty> = <x,y>.$$

令 $y=x$, 有 $|Tx|^2=|x|^2$. 双方开方,注意到范数恒为非负的,故有 $|Tx|=|x|$. 即是说 T 也是保范数的.

(b) 先证明 T 是 1-1 的. 一方面 T 自然地映 $x\in\mathbf{R}^n$ 为 $y=Tx\in\mathbf{R}^n$. 即,有一个 x,T 只有一个像;另一方面对于一个 $y\in\mathbf{R}^n$, 也只有一个原像. 因为设有两个 x_1, $x_2\in\mathbf{R}^n$ 均有 $y=Tx_1=Tx_2$,则 $0=Tx_1-Tx_2=T(x_1-x_2)$. 但是因为 T 是保范数的,所以 $|x_1-x_2|=|Tx_1-Tx_2|=0$, 即 $x_1=x_2$. 这就是说,如果 y 有原像存在,则原像必惟一. 另一方面原像一定是存在的,因为 $Tx=y$ 其实是 n 个未知数 $x_1,\cdots,x_n$ 的 n 个线性方程组成的方程组. 线性代数知识告诉我们,若相应的齐次方程组解只有 0 解,则非齐次方程组 $Tx=y$ 必可解. 上面的惟一性证明正说明 $Tx=0$ 只有零解,所以 $Tx=y$ 必可解. 二者综合即知 $Tx=y$ 对任意 $y\in\mathbf{R}^n$ 必有惟一解 x 存在,即是说 T 是 1-1 的.

上面我们作出了逆变换 T^{-1}. 即对任一个 $y\in\mathbf{R}^n$,$T^{-1}y=x\in\mathbf{R}^n$ 是存在的. 由于 T 是保范数的,所以对任意的 $y\in\mathbf{R}^n$ 有

$$|T^{-1}y|=|T(T^{-1}y)|=|y|.$$

这就是说 T^{-1} 也是保范数的. 再由(a)知 T^{-1} 也是保内积的.

1-8 (b)这个题目实际上就是平面几何中相似三角形的基本定理. 首先我们要注意,基底中不会含有 0 向量,所以一切 $x_i\neq 0$.

(1)先设 T 是保角的,现证对于一切 i, j 有 $\lambda_i=\lambda_j$. 我们只来证明 $\lambda_1=\lambda_2$.

取相应于它们的 x_1, x_2. 因为它们是基底的一部分,所以必

定线性无关而组成一个平面，T 则成为此平面到其自身的线性变换．于是这个问题表面上看是 n 维空间中的问题，实际上是 2 维平面上的问题(见图 A-1)．线性变换 T 变 $x_1 = \overrightarrow{OA}$ 为 $Tx_1 = \lambda_1 x_1 = \overrightarrow{O'A'}$，方向不变，变 $x_2 = \overrightarrow{OB}$ 为 $Tx_2 = \lambda_2 x_2 = \overrightarrow{O'B'}$，方向也不变，所以 $\angle AOB = \angle A'O'B'$．这里我们没有利用 T 的保角性，但用了 $\lambda_1, \lambda_2 > 0$，请读者画一个 $\lambda_1 > 0, \lambda_2 < 0$ 的图就知道了．

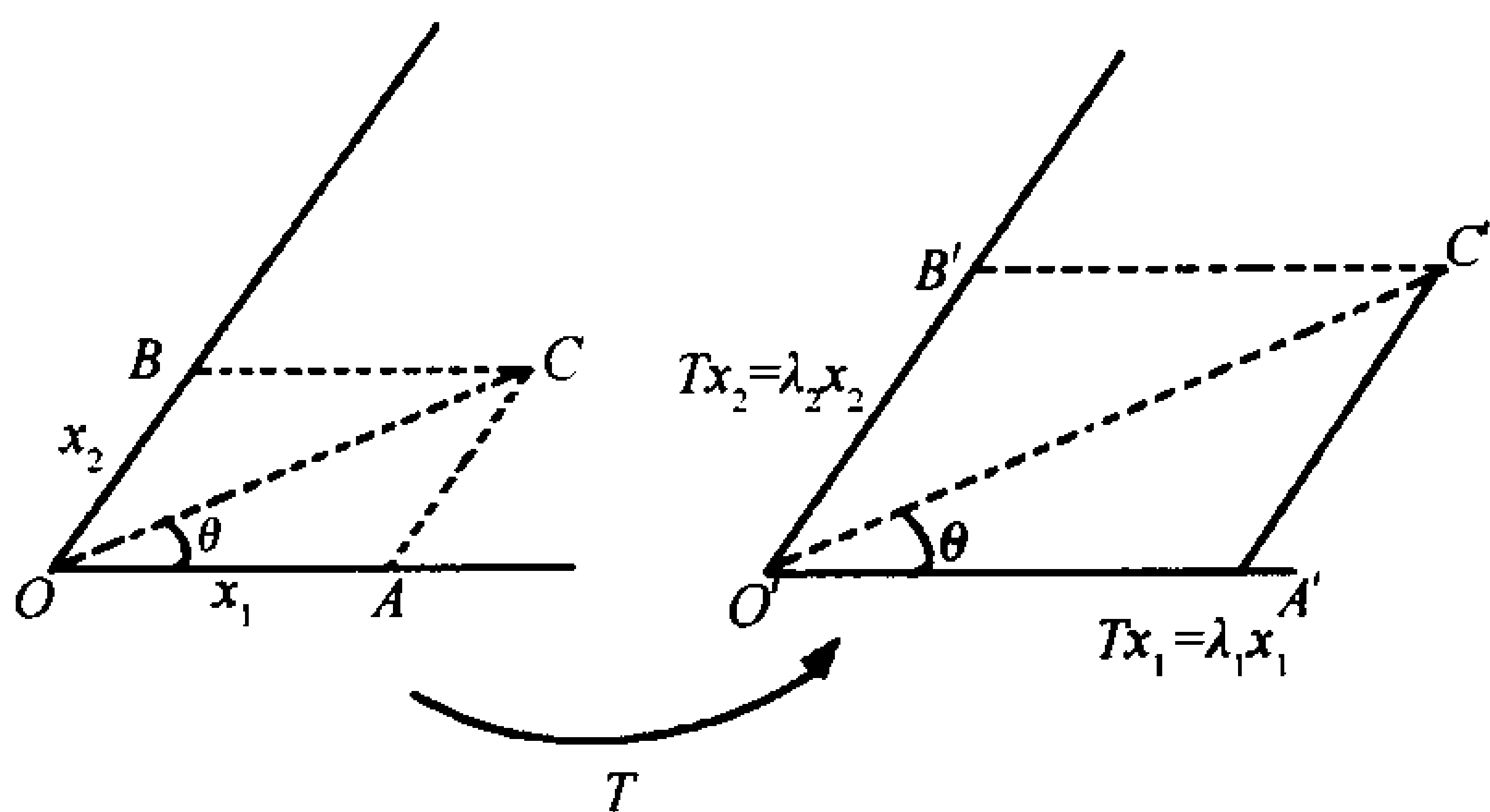


图 A-1

下面要利用 T 的保角性了．图上的 $\overrightarrow{OC} = x_1 + x_2$，而 $T(\overrightarrow{OC}) = Tx_1 + Tx_2 = \lambda_1 x_1 + \lambda_2 x_2 = \overrightarrow{O'A'} + \overrightarrow{O'B'} = \overrightarrow{O'C'}$．所以 $\overrightarrow{OC}$ 与 $\overrightarrow{OA}$ 之交角应等于 $\overrightarrow{O'C'}$ 与 $\overrightarrow{O'A'}$ 之交角：$\angle AOC = \angle A'O'C' = \theta$．现在来看 $\triangle AOC$ 与 $\triangle A'O'C'$．已经有了一个对应角相等，而 $\angle CAO = \pi - \angle AOB = \pi - \angle A'O'B' = \angle C'A'O'$．所以这两个三角形有两个相等的对应角，因此

$$\triangle AOC \backsim \triangle A'O'C',$$

由相似三角形的基本性质：

$$\frac{|\overrightarrow{O'A'}|}{|\overrightarrow{OA}|} = \frac{|\overrightarrow{A'C'}|}{|\overrightarrow{AC}|} = \frac{|\overrightarrow{O'B'}|}{|\overrightarrow{OB}|}.$$

但是 $\overrightarrow{O'A'} = T(\overrightarrow{OA}) = \lambda_1 \overrightarrow{OA}$，由于 $\lambda_1 > 0$，$|\overrightarrow{O'A'}| = \lambda_1 |OA|$（注意若$\lambda_1 < 0$ 则这个 结论要改），同理$|\overrightarrow{O'B'}| = \lambda_2 |\overrightarrow{OB}|$．代入上式即得

$$\lambda_1 = \lambda_2.$$

(2)现设 $\lambda_i = \lambda_j$对一切i,j均成立,而且所有 $\lambda_i > 0$. 这时想要利用上面简单的图形就不行了,因为这图仅适用于基底向量,而 T 对于它们的作用就简单地成为方向不变的伸缩. 于是现在我们取任意两个向量 $x,y \in \mathbf{R}^n$, 而由基底的假设

$$x = \sum_{i=1}^{n} a_i x_i, \qquad y = \sum_{i=1}^{n} b_i x_i,$$

因此

$$Tx = \sum_{i=1}^{n} a_i T x_i = \sum_{i=1}^{n} a_i \lambda_i x_i = \lambda \sum_{i=1}^{n} a_i x_i = \lambda x.$$

这里 λ 是$\lambda_i = \lambda_j$的公共值,它也是正的,同理

$$Ty = \sum_{i=1}^{n} b_i T x_i = \sum_{i=1}^{n} b_i \lambda_i x_i = \lambda \sum_{i=1}^{n} b_i x_i = \lambda y.$$

就是说,在 $\lambda_i = \lambda_j > 0$ 的条件下,T 对一切向量都是同样比例系数 λ 的伸缩. 因此

$$\frac{<Tx,\ Ty>}{|Tx|\ |Ty|} = \frac{\lambda^2 <x,\ y>}{|\lambda|^2 |x|\ |y|} = \frac{<x,\ y>}{|x|\ |y|},$$

这就是保角性.

$\lambda_i > 0$ 的条件不能去掉,因为有以下的反例. 考虑 $\mathbf{R}^2$,其中的基底 x_1, x_2 如图 A-2,而且夹角 $\alpha \neq \frac{\pi}{2}$, 我们定义

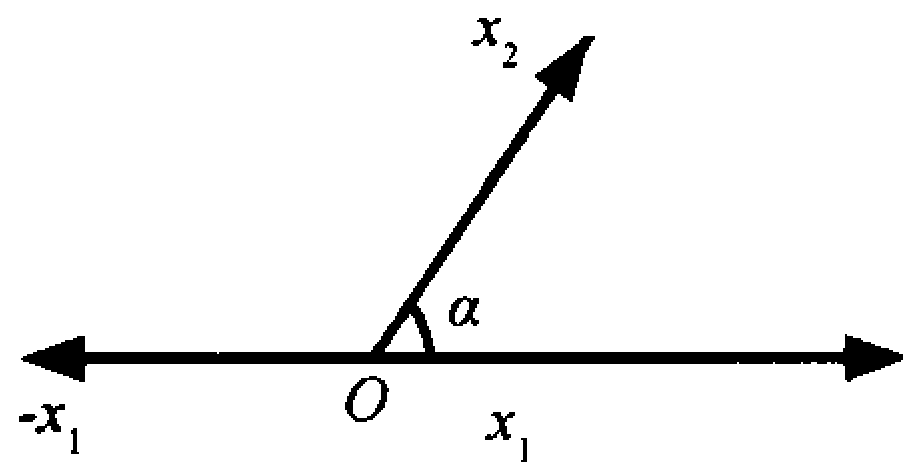


图 A-2

$$Tx_1 = -x_1, (\text{即}\lambda_1 = -1),$$
$$Tx_2 = x_2, (\text{即}\lambda_2 = 1).$$

对于一般的向量 $x = a_1 x_1 + a_2 x_2$, 则定义 $Tx = a_1 T x_1 + a_2 T x_2$. 这样得到一个线性变换 T. 它适合题中的条件,只除去 λ_1, λ_2 为正这个条件. 由图上可见 x_1, x_2 之夹角为 α, Tx_1, Tx_2 之夹角为

$\pi-\alpha$，二者不相等，因此这个 T 不是保角的.

1-10 令 $h=\begin{pmatrix}h_1\\ \vdots\\ h_n\end{pmatrix}$，$T=\begin{pmatrix}a_{11},\cdots,a_{1n}\\ \cdots\\ a_{n1},\cdots,a_{nn}\end{pmatrix}$，所以 $Th=\begin{pmatrix}\sum a_{1j}h_j\\ \vdots\\ \sum a_{nj}h_j\end{pmatrix}$.

因为

$$|h|^2=\sum(h_i)^2,\qquad |Th|^2=[\sum_i(\sum_j a_{ij}h_j)^2],$$

而

$$\sum_i(\sum_j a_{ij}h_j)^2\leqslant\sum_i C^2\sum_j h_j^2=(\sum_i C^2)|h|^2.$$

这里 $C=\max|a_{ij}|$. 所以由上式可得

$$|Th|^2\leqslant nC^2|h|^2=M^2|h|^2,\quad M=\sqrt{n}C,$$

因而得到结论. 注意本题中，用不同的方法可得不同的 M.

1-12 设有 $x\in\mathbf{R}^n$，利用 $<x,y>$（$\forall y\in\mathbf{R}^n$）定义一个 $\mathbf{R}^n$ 上的线性泛函

$$\varphi_x(y)=<x,y>\qquad(\forall y\in\mathbf{R}^n)$$

（因为这个泛函是由 x 定义的，故记为 φ_x），即得一个由 $\mathbf{R}^n$ 到 $\mathbf{R}^{n*}$ 的变换 $T:\mathbf{R}^n\to\mathbf{R}^{n*}$，$x\to\varphi_x$. T 是线性的易证，略去. 今证 T 是由 $\mathbf{R}^n$ 到 $\mathbf{R}^{n*}$ 上的变换，事实上，任给一个 $\Phi\in\mathbf{R}^{n*}$，则因为 $(\mathbf{R}^n)^*$ 的元就是 $\mathbf{R}^n$ 上的线性齐次式，故对任意 $y=(y_1,\cdots,y_n)$，有 $\Phi(y)=\sum_{i=1}^n\alpha_iy_i$，$\alpha_i$ 是实数. 令 $x=(\alpha_1,\cdots,\alpha_n)$ 即有

$$\Phi(y)=<x,y>=\varphi_x(y)$$

所以任意 $\Phi\in\mathbf{R}^{n*}$ 均在 T 的像中.

T 是 1-1 变换也易证. 一方面有一个 x，只有一个 $\varphi_x\in\mathbf{R}^{n*}$ 使

$$\varphi_x(y) = \langle x, y\rangle.$$

另一方面,对于任意 $\Phi \in \mathbf{R}^{n*}$ 又只有惟一的 x 适合 $Tx = \Phi$, 即 $\langle x, y\rangle = \Phi(y)$ ($\forall y \in \mathbf{R}^n$). 因为若有 x_1, x_2 均适合上式, 必有对一切 $y \in \mathbf{R}^n$,

$$\langle x_1, y\rangle = \langle x_2, y\rangle = \Phi(y) \text{ 即 } \langle x_1 - x_2, y\rangle = 0.$$

令 $y = x_1 - x_2$, 即有 $|x_1 - x_2|^2 = 0$, 亦即 $x_1 = x_2$.

1-14 任给一族开集 $\{U_\alpha\}$(我们写 α 而不写 n,表示允许有不可数多个开集 U_α), 若 $x \in \bigcup\limits_\alpha U_\alpha$, 则存在一个特定的 U_α(例如 U_1),使 $x \in U_1$ 而对 U_1 必存在一个开矩形 A, 使 $x \in A \subset U_1$, 但 $U_1 \subset \bigcup\limits_\alpha U_\alpha$, 则 $A \subset \bigcup\limits_\alpha U_\alpha$, 所以存在一个含 x 的开矩形 A 适合 $x \in A \subset \bigcup\limits_\alpha U_\alpha$, 因此 $\bigcup\limits_\alpha U_\alpha$ 是开集.

关于开集之交, 我们不妨只看两个开集 U_1, U_2 的特例, 如果 $x \in U_1 \cap U_2$, 则因 $x \in U_1$, 故有含 x 的开矩形 A_1 使 $x \in A_1 \subset U_1$. 同理又有一个含 x 的开矩形 A_2 使 $x \in A_2 \subset U_2$. 在平面情况下, 由图 A-3 可见一定有含 x 的矩形 $B \subset A_1 \cap A_2 \subset U_1 \cap U_2$, 所以 $U_1 \cap U_2$ 为开集. 对任意**有限多个**开集 $U_1, \cdots, U_n$ 上面的证明仍有效. 但对无穷多个 $\{U_\alpha\}$, 因为涉及 $\{A_\alpha\}$ 的"极限", 上面的方法就无效了. 但是方法无效不等于说结论不对, 因为还可能有其他方法. 这就需要找一个反例. 为简单起见, 我们只看 $n = 1$ 的情况: $\left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$对一切正整数 n 都是包含$[-1,1]$ 的开区间, 但是 $\bigcap\limits_n \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1]$ 是闭的而不是开集.

另一个情况是, 若 $U_1 \cap U_2 = \phi$, 本题结论仍是对的, 但空集为开不能用本书那样讲法来讨论, 请读者参看关于点集拓扑学的书籍.

另外, 请读者对一般的 $\mathbf{R}^n$, 不用上面的图直接证明本题.

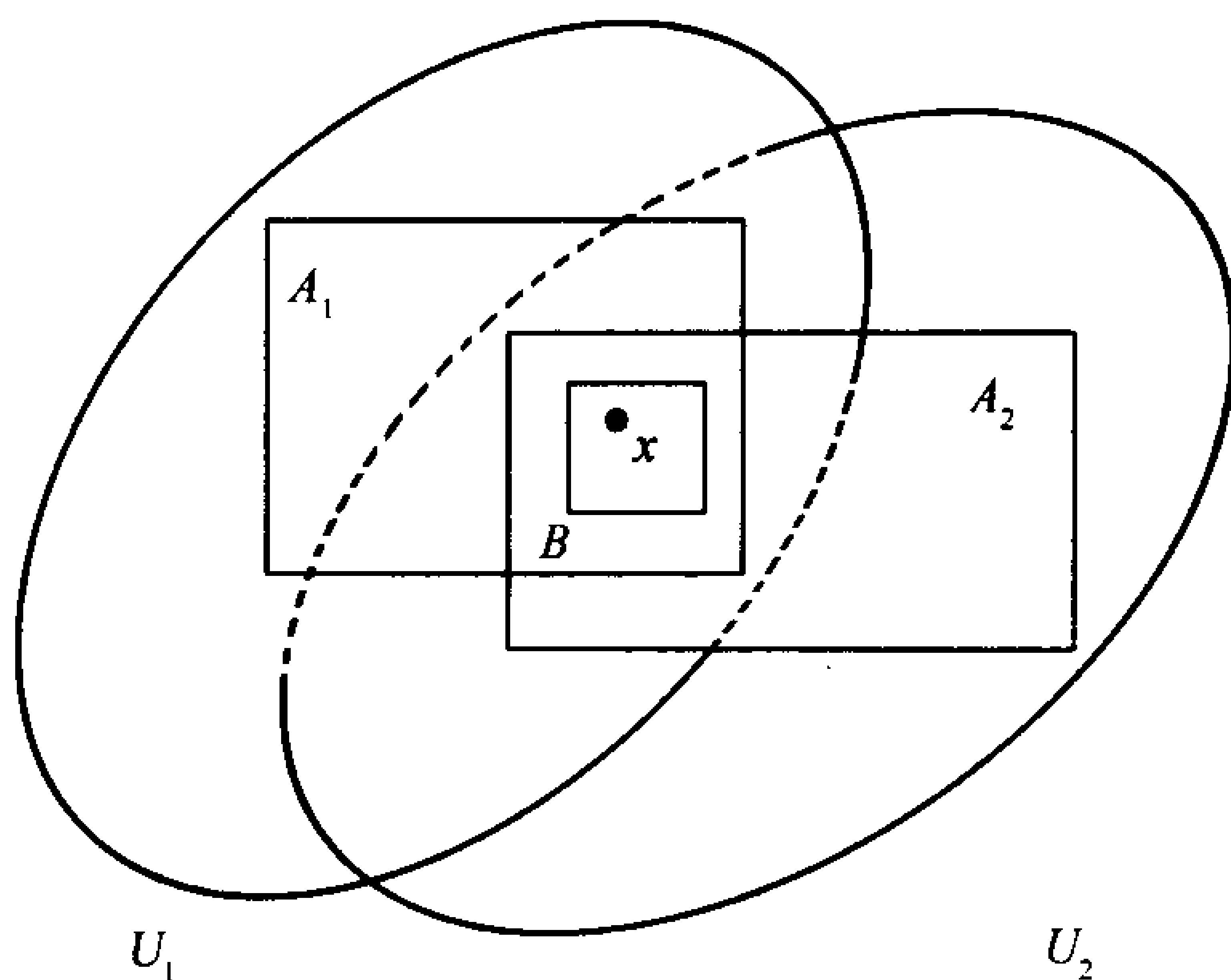


图 A-3

1-16 $\{x \in \mathbf{R}^n : |x| \leqslant 1\}$ 的内域是$\{x \in \mathbf{R}^n : |x| < 1\}$，外域是 $\{x \in \mathbf{R}^n : |x| > 1\}$，边界是 $\{x \in \mathbf{R}^n : |x| = 1\}$.

$\{x \in \mathbf{R}^n : |x| = 1\}$ 没有内域，外域是 $\{x \in \mathbf{R}^n : |x| > 1$ 或 $<1\}$，边界即其自身.

$\{x \in \mathbf{R}^n$，每个 x^i 是有理数$\}$ 没有内、外域，边界是全空间 $\mathbf{R}^n$.

1-17 取 $[0,1]$ 的两个有理数列 $\{x_n\}$ 和 $\{y_n\}$，使它们都是稠密的而且 $x_i \neq x_j$，$y_k \neq y_l$，$A = \{(x_n, y_n)\}$，即适合要求.

1-18 由 1-14 知 A 是开集，故内域为其自身，而 $\mathbf{R} - A$ 是 A 之外域与边界之并. 如果 $x \notin [0,1]$，则必有一个包含 x 的开区间 B 使 B 与 A 不相交，即 B 中之点均在 A 的外域中. 反过来，A 的外域中之点又一定在 $[0,1]$ 之外. 总之，A 的边界应是 $(\mathbf{R} - A) \cap [0,1] = [0,1] - A$.

1-20 设 $A \subset \mathbf{R}^n$ 是紧集，今证 A 必为有界闭集.

A 为有界集证明如下. 如果 A 是无界的，任给一个自然数 N，必有 $x_1 \in A$ 使 $|x_1| \geqslant N$. 因为 A 不能以 $|x_1| + 1$ 为界，故一定有一个 $x_2 \in A$，$|x_2| \geqslant |x_1| + 1$，因此 $|x_2 - x_1| \geqslant |x_2| - |x_1| \geqslant 1$. 仿此可以求出 $\{x_k\}$，使 $|x_{k+j} - x_k| \geqslant 1$，$j = 1, 2, \cdots$，$k = 1, 2, \cdots$.

以 A 中任一点 x 为心，$\frac{1}{2}$ 为"半径"作开矩形 U_x，则 $\{U_x\}$ 是 A 的开覆盖，这个开覆盖中不可能取有限个开覆盖 A，因为每一个 U_x 中至多有 $\{x_k\}$ 中的一个点，因此 A 不能为紧.

再证 A 为闭集，这一点可以参看习题 1-28 与 1-29.

1-23 令 f 的各个分量是 $f^1,\cdots,f^m$（这样的 f 只有 m 个分量），b 的分量是 $b^1,\cdots,b^m$. 由极限的定义知，对任一个 $\varepsilon>0$，存在 $\delta(\varepsilon)>0$，使当 $|x-a|<\delta(\varepsilon)$ 时 $|f(x)-b|<\varepsilon$. 但是 $f(x)-b$ 的分量是 $f^1(x)-b^1,\cdots,f^m(x)-b^m$，而且 $|f^i(x)-b^i|\leqslant|f(x)-b|$. 所以当 $|x-a|<\delta(\varepsilon)$ 时，$|f^i(x)-b^i|\leqslant|f(x)-b|<\varepsilon$，即 $\lim\limits_{x\to a}f^i(x)=b^i, i=1,2,\cdots,n.$

另一方面，很容易证明由 $\lim\limits_{x\to a}f^i(x)=b^i$，$i=1,2,\cdots,n$ 可得 $\lim\limits_{x\to a}f(x)=b$（略去），故本题得证.

本题与 1-24 合起来表明，研究由 A 到高维空间 $\mathbf{R}^m$ 的映射 f 的极限与连续性，可以归结为研究其每个分量的极限与连续性.

1-26 (a) 见图 A-4 自明，注意 $(0,0)\notin A$.

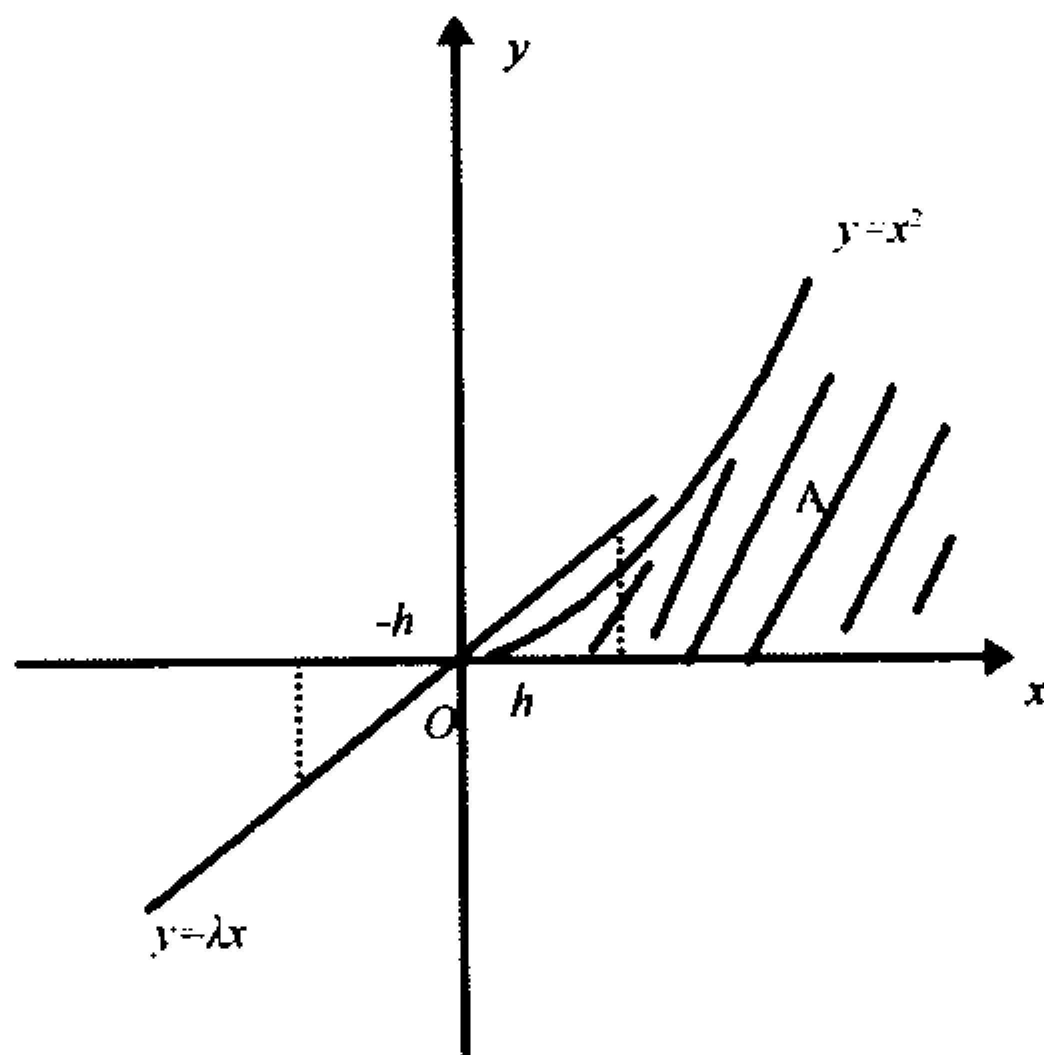


图 A-4

考虑直线 $y = \lambda x$，$\lambda = 0$ 时得到 $y = 0$. 此直线全部不在 A 中，而其右半是 A 的边界之一部分．$\lambda \neq 0$ 时，（图上画的直线 $\lambda > 0$)，取一个小区间$(-h,h)$，当$|h|$充分小时 $|\lambda h| \geqslant |h|^2$ ($h = 0$ 时等号成立)，这就表示直线 $y = \lambda x$ 位于 $x \in (-h,h)$ 上的一段(它自然以 $(0,0)$ 为内点)全在 $\mathbf{R}^2 - A$ 中．

(b) f 在 $(0,0)$ 处的不连续是明显的，但是 $f(th) = g_h(t)$ 是定义在上述直线上的函数．因为这条直线在 $(0,0)$ 附近（即当$|t|$充分小时）全在 $\mathbf{R}^2 - A$ 中，从而其上 $f(x) = 0$，即 $g_h(t) \equiv 0$. 当$|t|$充分小时，它当然在 $t = 0$ 处(即 $\mathbf{R}^2$ 中的 $(0,0)$ 点)连续．

1-28 因为 A 不是闭的，所以它至少有一个边界点 $x \notin A$，即 $x \in \mathbf{R}^n - A$. 令 $f(y) = 1/|y-x|$，当 $y \in A$ 时，$|y-x| \neq 0$，所以 $f(y)$ 当 $y \in A$ 时是连续的．但因在 A 之边界点 x 的任意小的含有 x 的开矩形中均有 $y \in A$ 存在，所以 $|y-x|$ 可以任意小，从而 $f(y)$ 是无界的．

1-29 先证明这样的 f 必是有界的．因为任取一点 x_0，$f(x)$ 既然在 x_0 连续，必有含 x_0 的一个开矩形 R_{x_0}，使在其上 $|f(x)-f(x_0)| < 1$，或 $|f(x)| < 1 + |f(x_0)|$ 对 R_{x_0} 成立．对 A 中每一点都可以作出一个类似的 R_x，所以 $\bigcup_x R_x$ 成为 A 的一个开覆盖．由 A 的紧性可知，$\bigcup_x R_x$ 一定有一个有限的子集 $\{R_{x_1},\cdots,R_{x_N}\}$ 仍然覆盖 A. 我们称之为 $\bigcup_x R_x$ 的一个子覆盖：$\bigcup_{k=1}^{N} R_{x_k} \supset A$. 于是对任一点 $x \in A$，必有一个 R_{x_i} 使 $x \in R_{x_i}$，从而 $|f(x)| < 1 + |f(x_i)|$. 在有限多个数 $|f(x_1)|,...,|f(x_k)|$ 中必有最大的一个记为 K，于是对一切 $x \in A$，$|f(x)| < 1 + K$. 即是说：**定义在紧集 A 上的连续函数 f_x 为有界的**．因此也一定有上确界 M 与下确界 m.

现在证明 $f(x)$ 必可达到 M 与 m. 以 M 为例，如果 $f(x)$ 不能达到 M，则 $M - f(x)$ 必定是 A 上的连续非零函数，从而 $1/[M - f(x)]$ 也在 A 上连续．但由上确界之定义，对任意 $\varepsilon > 0$ 一定存在一个 $\bar{x} \in A$ 使 $M > f(\bar{x}) > M - \varepsilon$. 因此 $0 < M - f(\bar{x}) < \varepsilon$ 从

而 $1/[M-f(\bar{x})]>\frac{1}{\varepsilon}$. 即是说 A 上有一个无界的连续函数：$1/[M-f(x)]$. 这与上面的结论是矛盾的. 所以本题得证.

由此可以再看习题1-20. 如果紧集 $A\subset\mathbf{R}^n$ 是非闭的，则由习题1-28，A 上一定有无界的连续函数，而这与本题矛盾. 因此这就完成了重要的习题1-20的证明.

请读者充分注意1-20，1-28，1-29这几个题目. 所有读过经典的多元函数微积分的读者都很容易模仿一元函数的方法，利用例如有界序列必有收敛子序列等等定理. 我们在这里完全没有涉及序列的问题，并不是仅仅因为现在的证法简单，更重要的是因为紧性是一个十分重要的，而且在一定程度上与序列问题相独立的概念. 尽管我们已经证明了，$\mathbf{R}^n$ 中的紧集就是有界闭集，但是“紧”与“有界闭”本质上是不同的，因此应该用不同方法处理. 我们给出这三个题目的详细证明目的就是提醒这一点. 读者可以参阅齐民友《重温微积分》第403～408页，高等教育出版社，2004.

1-30　取 $\xi_0=a<\xi_1<\cdots<\xi_n=b$ 使 $\xi_0\leqslant x_1<\xi_1<x_2<\cdots<x_n\leqslant\xi_n$，于是由 $f(x)$ 为增函数以及函数 f 在 x_i 点的振幅之定义有

$$o(f,x_i)<f(\xi_i)-f(\xi_{i-1}).\quad i=1,2,\dots,n.$$

相加即得本题结果.

2. 微　　分

2-4　(a)分两种情况. 首先设 $x=0$，则 $h(t)=f(t.\ 0)=f(0)=0$. 作为一个常值函数，它当然是可微的.

其次，若 $x\neq0$，则由 $f(tx)$ 之定义

$$h(t)=\begin{cases}|tx|\cdot g(\frac{tx}{|tx|}). & t\neq 0\\ 0 & t=0\end{cases}$$

当 $t\geqslant 0$ 时,$h(t)$ 是 t 的线性函数

$$h(t)=t|x|g(\frac{x}{|x|}),$$

所以一定可微. $t<0$ 时证法类似（利用 $g(-x)=-g(x)$）. $h(t)$ 在 $t=0$ 处的可微性请读者自行证明.

(b) 上面实际上是只在过(0,0)的任一直线段上看 $f(x)$，所以得到 $h(t)$ 可微. 现要在含(0,0)的一个完全的邻域中考虑 $f(x)$，这时就应该直接由定义，看能否找到一个线性变换 $Df(0)$ 使

$$\frac{|f(h)-f(0)-Df(0)h|}{|h|}\to 0, \tag{1}$$

这里 $h=(h_1,h_2)$ 是一个二维向量. 注意到 $f(0)=0$, $f(h)=|h|\cdot g(\frac{h}{|h|})$（应该令 $h\neq 0$. 为什么?). 分别考虑 $h=(h_1,0)$ 与 $(0,h_1)$ 的情况,利用 $g(1,0)=g(0,1)=0$,有 $f((h_1,0))=f((0,h_1))=0$. 因此对这种特殊的 h，必须取 $Df(0)=0$. 但 $Df(0)$ 如果存在必与 h 无关，所以如果 f 在 $0=(0,0)$ 处可微,必须有 $Df(0)=0$. 代回（1）式有 $\frac{|f(h)-f(0)|}{|h|}\to 0$, 但这是不可能的. 证毕.

2-6 考虑 $f(x,y)$ 在直线方程 $y=\lambda x$ 上的限制，可得

$$f(x,\lambda x)=\sqrt{|\lambda|}\cdot|x|,$$

然后再证明 $f(x,y)$ 在 $(0,0)$ 处不可微.

2-7

$$f(h)-f(0)=O(|h|^2).$$

$$\frac{|f(h)-f(0)-0\cdot h|}{|h|}\to 0.$$

2-8 把可微性定理(或定义)用到f上去，并将f分成两个分量分别考虑.

2-11 (a)利用链法则，令 $x+y=z$, 则

$$f(x,y)=F(z)\big|_{z=x+y},\qquad F(z)=\int_a^z g.$$

下面再利用 $x+y$ 是可微函数以及定理2-2 即得.

2-12 所谓双线性函数$f(x,y)$即对 x 与 y 分别为线性的函数. 如果把 x,y 各用分量来表示: $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_m)$, 当有 $f(x,y)=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}x_iy_j$, a_{ij} 为常数. 再注意到

$$|x_iy_j|\leqslant\sum_{k,l}|x_ky_l|\leqslant(\sum_{k=1}^{n}|x_k|^2)^{\frac{1}{2}}(\sum_{l=1}^{m}|y_l|^2)|^{\frac{1}{2}}\leqslant|(x\cdot y)|^2,$$

就易证明$\dfrac{|f(h,k)|}{|(h,k)|}\to0$. 其余部分自明.

但是以上的证明是利用 x,y 之分量（或坐标）来进行的，而微分学的结论应该是与坐标无关的. 本书重点之一在此. 因此，下面我们再讲一下如何不利用分量来进行证明. 为此直接回到定义，并且考虑 $Df(a,b)$. 它应该是由下式定义的:

$$\lim\frac{|f(a+h,b+k)-f(a,b)-Df(a,b)(h,k)|}{|(h,k)|}=0.$$

$Df(a,b)$是一个由 $\mathbf{R}^{m+n}$ 到 $\mathbf{R}^p$ 的线性变换. 上式中分子是 $\mathbf{R}^p$ 中的范数, (h,k) 是一个 $\mathbf{R}^{m+n}$ 向量, $|(h,k)|$是 $\mathbf{R}^{m+n}$ 范数. 若用分量表示，令 $h=(h_1,\ldots,h_m)$, $k=(k_1,\ldots,k_n)$ 则$(h,k)=(h_1,\ldots,h_m;k_1,\ldots,k_n)$ 而 $|(h,k)|^2=\sum_{i=1}^{m}|h_i|^2+\sum_{j=1}^{n}|k_j|^2$.

我们首先要提到一个**极重要而又极易引起误解的结论：若 $L:\mathbf{R}^l\to\mathbf{R}^p$ 是一个线性函数，则其“导数” DL 就是它自己:**

$$(DL)(h)=L(h).\tag{1}$$

它的证明很简单．既然 L 是线性函数，它当然就是一个线性变换：$\mathbf{R}^l \to \mathbf{R}^p$，而“导数”D$L$ 按定义同样是由 $\mathbf{R}^l$ 到 $\mathbf{R}^p$ 的线性变换，所以(1)之双方有意义．另一方面，L 为线性函数，所以对任意 h 有

$$L(a+h)-L(a)=L(h),\qquad a,h\in\mathbf{R}^l.$$

所以

$$\frac{|L(a+h)-L(a)-L(h)|}{|h|}=0 \text{ 当然也趋于 } 0.$$

将此式与“导数”DL 之定义比较，可见 L 就是 DL，从而(1)式成立．

说它容易引起误会，是因为我们在一元函数微积分学中学过：若一函数 $f(x)$ 之导数 $\frac{\mathrm{d}f(x)}{\mathrm{d}x}=f(x)$，则 $f(x)=Ce^x$，而不是线性函数．这个方法在于，在学一元函数微积分时，我们有 $\frac{\mathrm{d}f(x)}{\mathrm{d}x}=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$. 即是令 $\Delta x\to 0$，而 x 不动．但谈及 $\frac{\mathrm{d}f(x)}{\mathrm{d}x}=f(x)$时，是将等式两边看作是 x 的函数，而 Δx 在取极限后已经消失了．与现在的讲法比较，我们是用 h 来表示 Δx 的，它是一个 $\mathbf{R}^l$ 向量，而 a 则相应于 x. 所以本书的讲法很清楚，a 与 h 是分得很清楚的，而且 a 一直是固定的．如果一定要与 $\frac{\mathrm{d}f(x)}{\mathrm{d}x}=f(x)$ 比较，左方 D$f(a)$是一族线性变换作用于 h，而以 a 为参数，右方是函数 f 在 a 点的值，二者无法比较．这一点正是微分概念的精华，参见齐民友《重温微积分》74－78 页（特别是 78 页）．

(b) 现在回到本题并且看双线性函数（亦即双线性变换）$f(x,y):\mathbf{R}^n\times\mathbf{R}^m=\mathbf{R}^{m+n}\to\mathbf{R}^p$. 由导数之定义，我们应该看

$$\begin{aligned}f(a+h,b+k)-f(a,b)=&[f(a+h,b+k)-f(a,b+k)]\\&+[f(a,b+k)-f(a,b)].\end{aligned}$$

先看第2项，这里 a 是固定的，而 $f(a,y)$ 是 y 的线性函数，所以由上所述 $f(a,b+k)-f(a,b)=f(a,k)$. 再看第1项，根据同样的理由

$$f(a+h,b+k)-f(a,b+k)=f(h,b+k).$$

再利用一次 f 对于后一变量为线性性质有

$$f(a+h,b+k)-f(a,b+k)=f(h,b)+f(h,k).$$

回到定义即得

$$f(a+h,b+k)-f(a,b)=f(k,b)+f(a,k)+f(h,k).$$

前两项合并成为关于 (h,k) 的线性式，而最后一项如果用 (k,h) 之分量来写应是 $\sum_{i=1}^{n}\sum_{j=1}^{m}C_{ij}h_i k_j$. 因此

$$|f(h,k)\leqslant\sum_{i=1}^{n}\sum_{j=1}^{m}C|h|\cdot|k|\leqslant M|(h,k)|^2,$$

故由导数的定义有

$$\mathrm{D}f(a,b)(h,k)=f(h,b)+f(a,k).$$

把 h,k 分别改为 x,y 即得本题所求证.

(c)上面的结果是十分重要的，因为数学中许多运算都具有双线性性质. 最简单的如两个独立变量 $x\in\mathbf{R}$ 与 $y\in\mathbf{R}$ 之积 $p(x,y)=x\cdot y$, 我们都说它是一个二次齐性函数. 但若把 x 和 y 分开来看，则视 p 为一个变换，应有

$$p:\quad \mathbf{R}\times\mathbf{R}\to R,\ (x,y)\mapsto x\cdot y$$

这恰好是上面讲的 $m=n=p=1$ 的最简单的双线性变换. 故由(b)有

$$\mathrm{D}p(a,b)(x,y)=ay+bx,$$

这就是定理2-3的(5).

2-13　(a)另一个十分重要的“函数”是两个 $\mathbf{R}^n$ 中的向量 x 与 y 的内

积（或称数量积）：

$$\mathrm{IP}(x,y)=\langle x,y\rangle.$$

从上题的观点看来，它也是一个双线性变换

$$\mathrm{IP}:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}.\ (x,y)\mapsto <x,y>.$$

所以由 2-13 题可知

$$\mathrm{D(IP)}(a,b)=\langle a,y\rangle+\langle x,b\rangle.$$

这是一个线性变换 $\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$.

$(\mathrm{IP})'(a,b)$是这个线性变换的矩阵．如果我们把 $\mathbf{R}^n\times\mathbf{R}^n$之元素写成一个竖向量（列向量），即 $2n\times 1$ 矩阵，例如

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x_1\\\vdots\\x_n\\y_1\\\vdots\\y_n\end{pmatrix},\text{则}\langle a,y\rangle=(0,\cdots,0;a_1,\cdots,a_n)\begin{pmatrix}x_1\\\vdots\\x_n\\y_1\\\vdots\\y_n\end{pmatrix}.$$

这里我们把横向量(行向量)$(0,\cdots,0;a_1,\cdots,a_n)$看成一个 $1\times 2n$ 矩阵．上式就是这个 $1\times 2n$ 矩阵作用于一个 $2n\times 1$ 矩阵，其结果是一个 1×1 矩阵，即一个实数．用同样的方法处理 $\langle x,b\rangle$，又可得另一个 $2n\times 1$ 矩阵 $(b_1,\cdots,b_n;0,\cdots,0)$．而

$$(\mathrm{IP})'(a,b)=(b_1,\cdots,b_n;a_1,\cdots,a_n)=\begin{pmatrix}b_1\\\vdots\\b_n\\a_1\\\vdots\\a_n\end{pmatrix}^{\mathrm{T}}.$$

2-14 则把上面讲的双线性变换推广为重线性变换.

2-15 我们都习惯了如何定义行列式. 但是对于它更深刻的看法是认为一个 n 阶行列式是一个 n 重线性变换. 重线性是很容易证明的. 因为如 2-14 那样, 我们认为 $x_j \in E_j$ 是一个 n 维列向量

(即对一切 $j, E_j \in \mathbf{R}^n$): $x_j = \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix}$, 而令

$$\det(x_1, \cdots, x_n) = \begin{vmatrix} x_{11} & & & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & & & x_{nn} \end{vmatrix},$$

它的元 x_{ij} 有两个指标 i 与 j. i 表示所在的行, j 表示所在的列. 于是重线性就归结为非常简单的行列式性质. 以第一列为例, 如果

$$x_1 = \lambda y_1 + \mu z_1,\ y_1 = \begin{pmatrix} y_{11} \\ \vdots \\ y_{n1} \end{pmatrix},\ z_1 = \begin{pmatrix} z_{11} \\ \vdots \\ z_{n1} \end{pmatrix},$$

而 λ, μ 为数. 则

$$\begin{aligned} \det(\lambda y_1 + \mu z_1, x_2, \cdots, x_n) &= \begin{vmatrix} \lambda y_{11} + \mu z_{11} & & x_{1n} \\ \vdots & \vdots & \vdots \\ \lambda y_{n1} + \mu z_{n1} & & x_{nn} \end{vmatrix} \\ &= \lambda \begin{vmatrix} y_{11} & & x_{1n} \\ \vdots & \vdots & \vdots \\ y_{n1} & & x_{nn} \end{vmatrix} + \mu \begin{vmatrix} z_{11} & & x_{1n} \\ \vdots & \vdots & \vdots \\ z_{n1} & & x_{nn} \end{vmatrix} \\ &= \lambda \det(y_1, x_2, \cdots, x_n) + \mu \det(z_1, x_2, \cdots, x_n). \end{aligned}$$

至于“导数”的求法则与2-12(b)一样.

2-19 在求 $D_2f(1,y)$ 时，是对第2个变元求导，就是说第1个变元不变，始终为1. 所以不妨先在 $f(x,y)$ 中令 $x=1$，然后再对所得的 y 的函数求导数. 注量到 x^y 当 $x=1$ 时恒有 $x^y|_{x=1}=1^y\equiv 1$. 又 $\ln 1=0$. 因此 $f(1,y)=1+0=1$，而问题解决.

2-22 (a)注意，A 现在是 $\mathbf{R}^2$ 平面除去一个割口：正 x 轴（包括原点），而区域的几何性质在研究微分问题时起了重大作用. 我们讲函数可微性首先是讲它在一点的某领域中的可微性，这个领域在本书中时常是一个（开）矩形，在许多其他书上则是一个（开）球，这都是最简单的几何形状. 例如我们都非常熟悉的定理：若在 $A\subset\mathbf{R}^1$ 上 $f'(x)=0$，则 $f(x)=$ 常数. 其实这样想就犯了大错. 因为我们不能说，当 $f(x)=$ 常数时有 $f'(x)=0$，故有以上结论. 这是把定理与逆定理混为一谈了. 正确的证法是利用拉格朗日公式：若 $f(x)$ 在 $[a,b]$ 连续，且在 $[a,b]$ 上可微，则必存在一个 $c\in(a,b)$，使

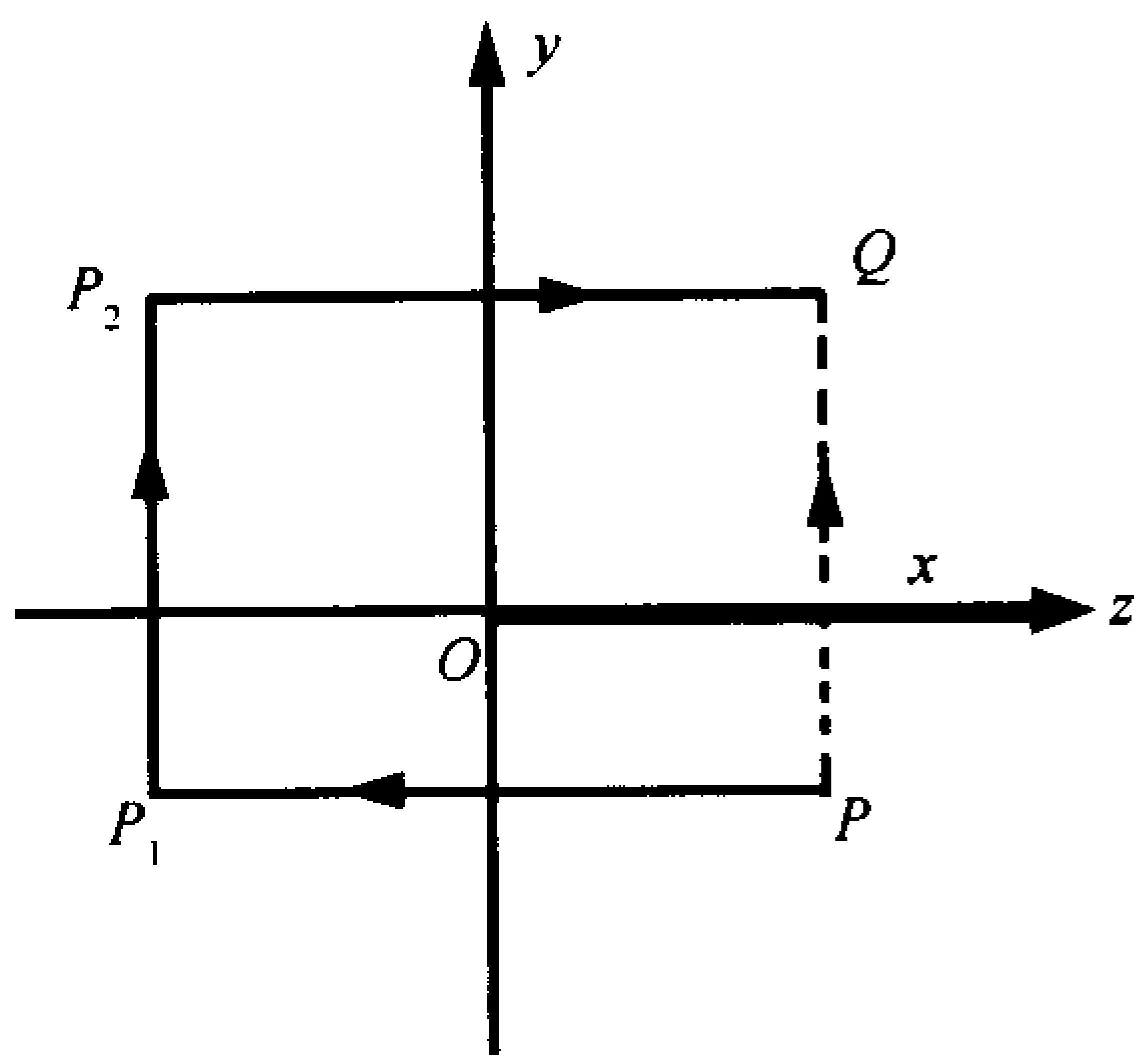


图 A-5

$$f(b)-f(a)=f'(c)(b-a).$$

这里时常被人们忽视的是：我们讲的是一个区间上的函数. 而在 $\mathbf{R}^1$ 中，区间（开或闭）是最简单的几何图形. 因此，我们只能说，若 $f(x)$ 定义在 $A\subset\mathbf{R}^1$ 上，而且 A 是一个区间，则由 $f'(x)=0$ 可得 $f(x)=$ 常数. 如果 A 不是一个区间，则这个结论是不成立的. 下面是反例. 设 $A=[0,1]\cup[2,3]$（两个互相分离的区间之并），而 $f'(x)=0$ 于 A 上，如图 A-6 则很可能

$$f(x)=\begin{cases}c_1,\ x\in[0,1]\\ c_2,\ x\in[2,3]\end{cases}\quad c_1\neq c_2.$$

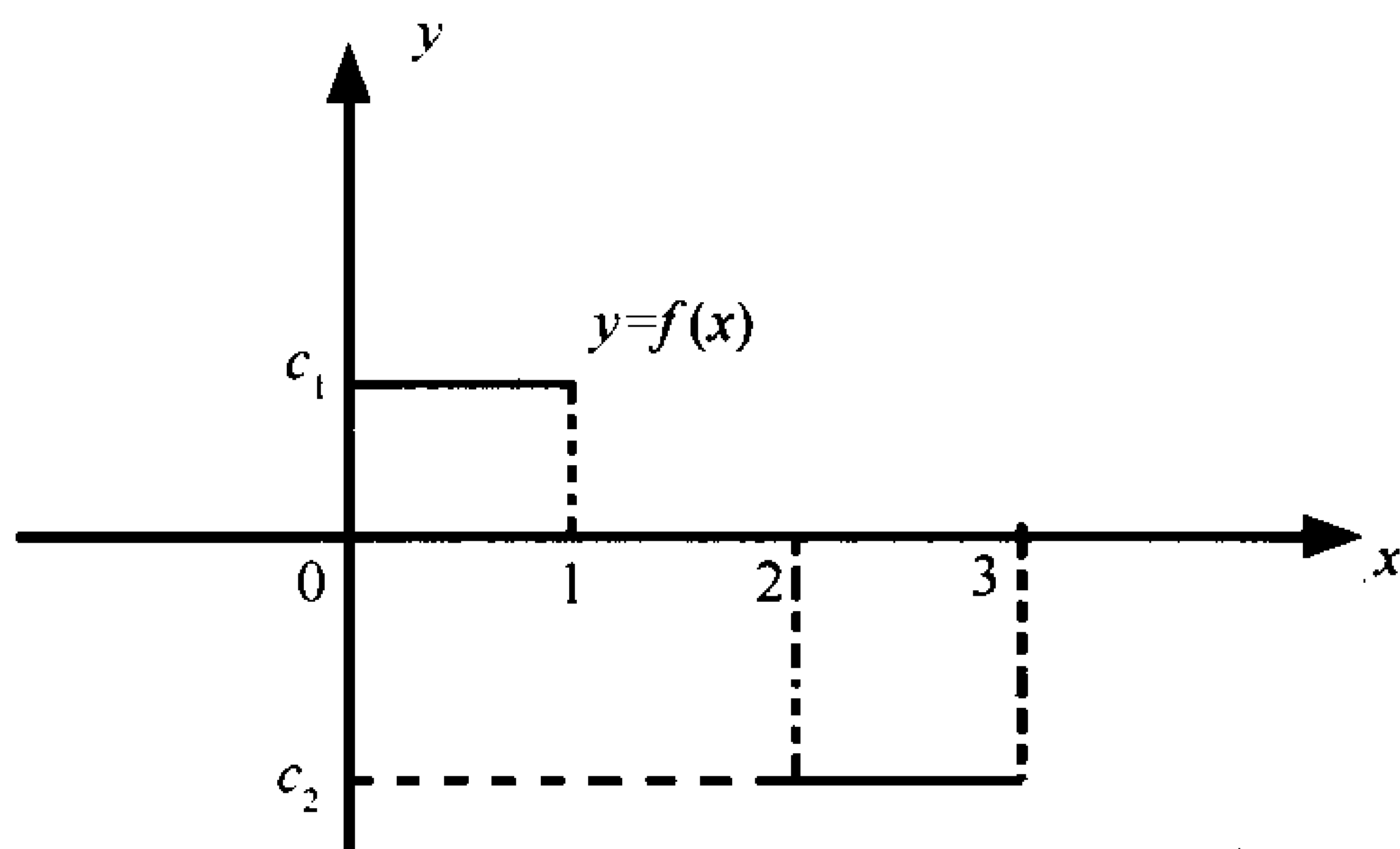


图 A-6

这样的$f(x)$时常称为“局部常值”函数．

所以本题中有一个提示：“注意，A 中任两点可用一串直线段相连”．因此如果我们要证明$f(x,y)$在 P,Q 两点之值相同，不能直接用图上的虚线连接 P,Q 两点，并视$f(x,y)$为此虚线上的 y 的函数，再利用上面的结论．因为它不是区间，而是虚线除去一点．因此有反例表明$f(x)$最多只能是局部常值函数．所以，正确的做法，是用折线 PP_1,P_1P_2,P_2Q. 它们都是区间，因此可以应用拉格朗日定理，而得 $f(P)=f(P_1)=f(P_2)=f(Q)$. 所以$f(x)$在 A 上取常值．

这里的关键问题是 A 中任两点可用折线连接起来而上面的反例则不行，这个性质与 A 的所谓连通性有关．可以说，经典的微积分教材中都只讲到实数系的完备性(柯西准则等)，而紧性则被“有界闭”掩盖了，连通性完全不谈．其实，所谓连续函数的“基本性质”中，我们已讲了有界性，可以达到最大最小值，它们都是紧性的表现．还有“有界闭区间上的连续函数必为一致连续”也是紧性问题．但是中值定理的关键则是连通性问题．在本书中类似本题这样涉及连通性的地方很多，突出的是第 4 章的定理 4-11. 它是极重要的，我们要特别领会其中

的思想．关于这一类问题仍请读者参看《重温微积分》一书第6章中关于紧性与连通性的讨论．

(b)请看(a)中的反例以及不能应用拉格朗日定理的说明．

2-25 这个例子很简单，但是非常重要．下一题就是讲如何应用它来构造“具有紧支集的无穷可微函数”．所谓支集就是使该函数不为0的点之集合再加上这个点集的边界点之集合．这句话有点绕，如果粗略一点说，具有紧支集的无穷可微函数就是那些在某个有界闭集之外恒为0的无穷可微函数，而且因为它们是连续的，所以在此有界闭集的边界上也必为0. 这与我们习惯了的所谓初等函数大不相同：那些函数都是只在孤立点上为0，甚至在实变量情况下恒不为0(如e^x)．但是这类函数太重要了，在所谓“现代分析”中，它们占有突出的地位．因其重要，所以有一个专门的记号：C_0^∞．例如本书第3章的定理3-11介绍所谓单位分解，就是以它(具体说是以习题2-26)为基础的．问题在于究竟有没有这种函数存在?这个题目就是做出这种函数的一种最常见的“原材料”，下一题再讲怎样用这种原料构造出更多有用的C_0^∞函数，而它的进一步的变化还更加丰富多彩．有兴趣的读者可以参看《重温微积分》第3章，特别是132～136页．

因为此题重要，我们再介绍另一种更清楚的证明，其基础是以下的

引理　函数

$$f(t)=\begin{cases}e^{-1/t}, t>0\\0, t\leqslant 0\end{cases}\tag{1}$$

是$\mathbf{R}^1$上的C^∞函数，而且对所有非负整数j, $f^{(j)}(0)=0$.

证　在$t>0$时$f(t)$无穷可微．在$t\leqslant 0$时$f(t)\equiv 0$，故不但是无穷可微而且各阶微商均恒为0. 因此，为证此引理，只须证明$\lim\limits_{t\to 0+}f^{(j)}(t)=0$即可．

注意到当 $t>0$ 时 $e^{1/t}=\sum_{m=0}^{\infty}\frac{1}{m!}(\frac{1}{t})^m$ 为一正项级数．所以对一切正整数 m 有 $e^{1/t}\geqslant\frac{1}{m!t^m}$，从而 $e^{-1/t}\leqslant m!t^m$．因此，当 $t>0$ 时

$$|f(t)|\leqslant m!t^{m+1}. \tag{2}$$

现在看 $f(t)$ 在 $t=0+$ 处的各阶（右）导数．易见，$t>0$ 时

$$f'(t)=\frac{1}{t^2}e^{-1/t},\text{所以}\lim_{t\to0+}f'(t)=0.$$

$t<0$ 时 $f'(t)\equiv0$．$\lim\limits_{t\to0-}f'(t)=0$．从而 $f'(t)$ 在 $t=0$ 时连续且 $f'(0)=0$，我们用数学归纳法证明，$f^{(j)}(0)=0$．事实上，用数学归纳法可得，$t>0$ 时

$$f^{(j)}(t)=P(\frac{1}{t})e^{-1/t}.$$

P 是 $2j$ 次多项式．在（2）式中取 $m=2j$，即有

$$|f^{(j)}(t)|\leqslant C|t|. \tag{3}$$

当 $t<0$，$f^{(j)}(t)\equiv0$．故 $\lim\limits_{t\to0-}f^{(j)}(t)=0$．由（3）式 $\lim\limits_{t\to0+}f^{(j)}(t)=0$．故当补充定义 $f^{(j)}(0)=0$ 后即得引理之证．

这个 $f(t)$ 并不具有紧支集，其支集为 $\{t\geqslant0\}$．

令 $t=|x|^2$ 即得本题．

2-26　则利用本题做 C_0^∞ 函数．其中（d）比较麻烦．我们通常的做法是利用所谓磨光技巧，这里无法解释．读者可以参看上面列的参考书．

2-31　此题的特点是：当研究偏导数时，可以限制自变量只在某一直线上变动．例如本题中既然是讨论 $D_\lambda f(0,0)$，即可限制 x 只在过 $(0,0)$ 而与 x 轴平行的直线上变化，即只看 $y=0$ 时的 $f(x,y)|_{y=0}=f(x,0)$．但 $y=0$ 不在 A 中，所以 $f(x,y)\equiv0$（对一切 x 成立），本题得证．习题 1-26 中，(x,y) 应可设一切路径

趋向 (0,0) 才能考虑 $f(x,y)$ 在 (0,0) 的连续性，而偏导数与方向导数则不要求这样，本题的含意在此．

2-35 这里请读者注意，f 之定义域是 $\mathbf{R}^n$. $\mathbf{R}^n$ 有一个特点，即原点与任意 x 点的连线仍在 $\mathbf{R}^n$ 中．这句话似乎多余，但若把 $\mathbf{R}^n$ 改成 A：令 A 中一切点 x 与其某一点(设为0)的连线，仍在 A 中，则本题似乎全然不足道，因为

$$h'_{\lambda}(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(tx) = x^1\mathrm{D}_1 f(tx) + \cdots + x^n\mathrm{D}_n f(tx).$$

沿连结0（即 $t=0$）与 x（即 $t=1$）的线段积分，有

$$f(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}f(tx) = \sum_{i=1}^n x^i\int_0^1 \mathrm{D}_i f(tx)\,\mathrm{d}t = \sum_{i=1}^n x^i g_i(x).$$

这里 $g_i(x)=\int_0^1 \mathrm{D}_i f(tx)\,\mathrm{d}t$. 以上推导中当然用了 $f(0)=0$，但这并不重要．因为当它不成立时，我们仍然可以得出

$$f(x) = f(0) + \sum_{i=1}^n x^i g_i(x). \tag{4}$$

而真正重要的是，“积分区间”即连结 0 与 x 的线段，若它不含于 A 中，积分就没有意义了．可见问题的关键仍在 A 的几何性质．具有这种性质的 A，称为关于 0 点的星形域．这与问题 2-23 很有类似之处，也请参看第 4 章“向量场与微分形式”一节．

本题的结论，特别是(4)式，在许多书上称为阿达玛引理，是一个很有用的结果．

2-36 请与定理 2-11（反函数定理）相比较．仔细分析其证明就可以看到，并不必去讨论 f 在整个定义域 $\mathbf{R}^n$ 上的反函数，而只讨论 $f:V\to W$. 因此对于 $f:A\to\mathbf{R}^n$，也不必讨论 f 在整个 A 上之性质，而只讨论包含 a 的一个开集 V（称为 a 的开邻域）. 这样 2-11 定理就可以几乎逐字地搬到这里，可令 $V=A$，$W=f(A)$. 由 2-11 定理知 $f^{-1}:f(A)\to A$ 也是连续的．但连续函数必定是值域中

的开集之原像，仍为开集．现在 A 是值域中的开集了，$f(A)$ 是 A 在 f^{-1} 下的原像，所以也是开集．$f(B)$ 为开集证明亦同．

2-37 (a)我们最好是从几何角度来看它的意义．在直观上很清楚，取一个适当的值 α，$f(x,y)=\alpha$ 表示一条曲线，如果 (x_α,y_α) 是此曲线上一点，则 f 恰好是把 $\mathbf{R}^2$ 的这一点映为 $\mathbf{R}^1$ 中的 α. 因此除非这条曲线"蜕化"成一个点 α，则 α 的原像必不止一点，从而 f 不可能是 1-1 的．由此可见，问题在于如何保证它是一条不蜕化的曲线．

原作者的提示的用意即在于此．这个提示建议考虑一个由 (x,y) 平面到 (z,w) 平面的变换

$$T:\quad \begin{cases} w = f(x,y) \\ z = y. \end{cases} \tag{1}$$

如果设 $D_1 f(x,y) \neq 0$ 于 A 中，则在其中

$$T' = DT = \begin{pmatrix} D_1 f & D_2 f \\ 0 & 1 \end{pmatrix}, \tag{2}$$

而 $\det T' \neq 0$，因此由反函数定理 L 在 A 中的一段必与 (z,w) 平面上过 (α,y_0) 的一个直线段 $1-1$ 对应．但是 f 是把 (x,y) 平面映到 $\mathbf{R}^1$ 上的，从图 A-7 来看可以认为是先做一个变换 T，把曲线 L"拉直"，再做一个沿 y 轴方向的投影 Pr.（我们时常就记为 $f = Pr\circ T$）．这一"拉直"就把直线段变成了图右边上的 P 点，因此它一定不是 1-1 的．

在这里，f_x,f_y 不同时为 0 是很重要的．看一个例子：

$$f(x,y) = 0\cdot x + 0\cdot y \equiv 0.$$

它当然仍然破坏了 1-1，但与上面讲的情况不同：上面是把一个 2 维的区域变为 1 维的——维数下降 1，现在是下降 2 成为 $2-2$ 维（即 0 维）的几何图形：一点 0.

(b)如果理解了(a)的几何意义，就知道应该考虑

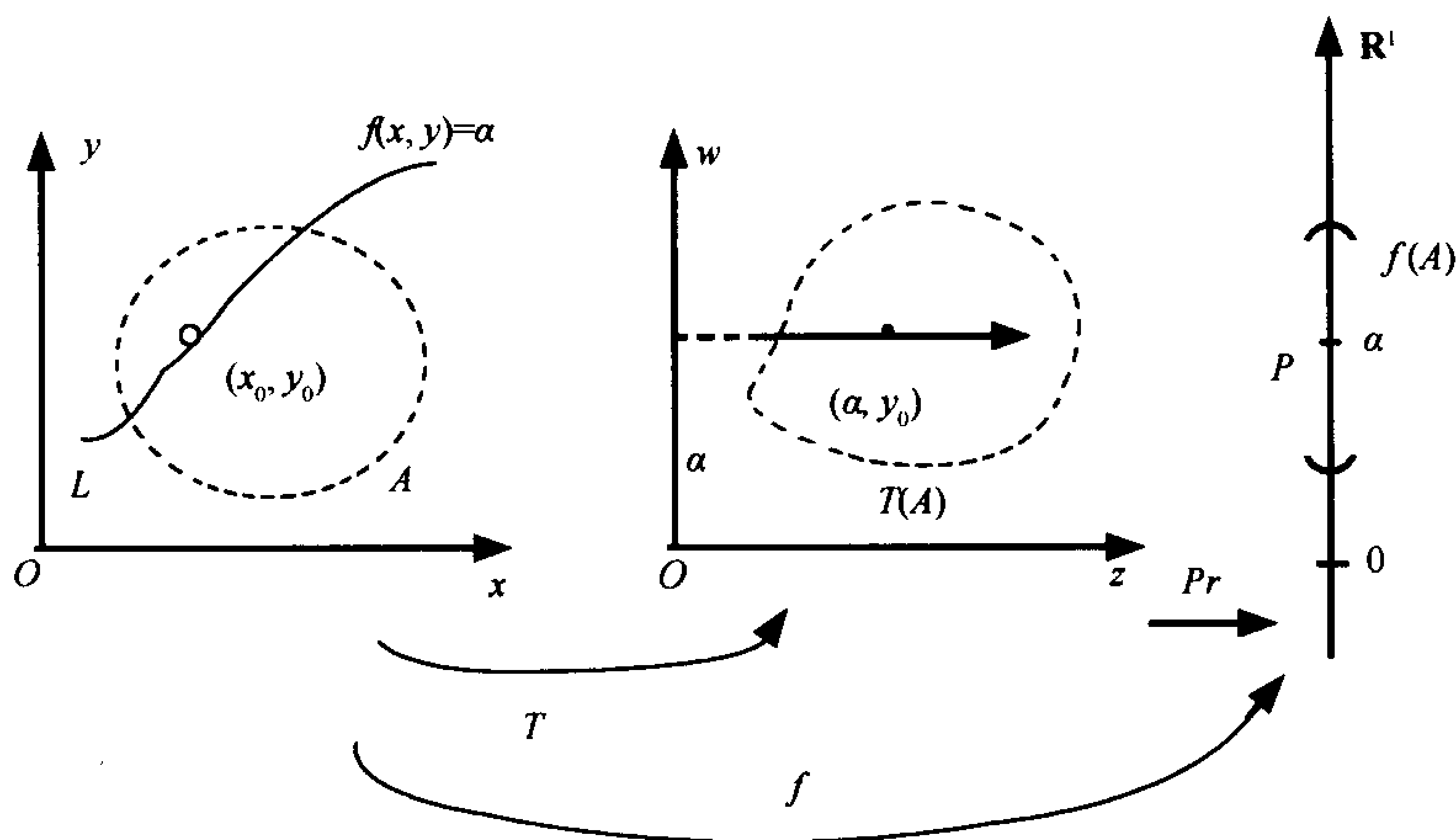


图 A-7

$$f' = \begin{pmatrix} \dfrac{\partial f_1}{\partial x^1} \cdots \dfrac{\partial f_1}{\partial x^n} \\ \dfrac{\partial f_m}{\partial x^1} \cdots \dfrac{\partial f_m}{\partial x^n} \end{pmatrix} \tag{3}$$

它没有行列式，但是有秩，它的最大可能的秩是 m，即是说需要在“最大秩条件”下研究它．其方法仍然是先造成一个 T，再做一个投影 Pr 使 $f = Pr \circ T, T$ 是 1-1 的，是“拉直变换”，而投影 Pr 破坏了 1-1 性．

由此还可想到，$m > n$ 时大概也不行．详见上面列的参考书．还可想到，若 f 是 1-1 的、连续可微的，必有 $m = n$. 这称为维数的不变性．可是如果只设 f 为连续，则它是一个很难的问题．

3. 积　　分

3-3 (a)因为在分法 P 的任一个子矩形 S 上同时有 $f \geqslant m_S(f), g \geqslant$

$m_S(g)$，所以$f+g \geqslant m_S(f)+m_S(g)$. 而这个不等式的右方是$f+g$在$S$上的一个下界. 由下确界的定义有

$$m_S(f)+m_S(g) \leqslant m_S(f+g).$$

选一个适当的分法P，可以使$\sum\limits_{S\in P} m_S(f)\cdot v(S) \geqslant L(f,P)-\varepsilon$，$\sum\limits_{S\in P} m_S(g)\cdot v(S) \geqslant L(g,P)-\varepsilon$，这里的$\varepsilon>0$是一任意正数. 因此对这个$P$

$$\begin{aligned} L(f,P)+L(g,P)-2\varepsilon &\leqslant \sum_{S\in P} m_S(f)\cdot v(S)+\sum_{S\in P} m_S(g)\cdot v(S) \\ &\leqslant \sum_{S\in P} m_S(f+g)\cdot v(S) \leqslant L(f+g,P). \end{aligned}$$

但是ε是任意的，令$\varepsilon\to 0$即有

$$L(f,P)+L(g,P) \leqslant L(f+g,P). \tag{1}$$

同理

$$U(f+g,P) \leqslant U(f,P)+U(g,P). \tag{2}$$

(b)所谓一个函数可积就是说其下和对分法P的上确界与上和对分法P的下确界相等，其公共值就称为此函数的积分. 现在因为f与g分别可积，所以可以假设有一个**共同的**分法P使$L(f,P) \geqslant \sup\limits_P L(f,P)-\varepsilon$，$L(g,P) \geqslant \sup\limits_P L(g,P)-\varepsilon$. $\varepsilon>0$是任意给定的. 代入(1)有

$$\sup_P L(f,P)+\sup_P L(g,P)-2\varepsilon \leqslant L(f+g,P),$$

此式左方已与P无关，而右方则是对某一个P而言的. 所以可以把右方的P改写成Q，"帮助"我们"忘记"它"曾经"与左方的P相同. 在此式右方对Q取上确界，然后再注意到ε之任意性，即有

$$\sup_P L(f,P)+\sup_Q L(g,P) \leqslant \sup_Q L(f+g,Q). \tag{3}$$

同理

$$\inf_{Q'} U(f+g,Q') \leqslant \inf_{Q'} U(f,Q') + \inf_{Q'} U(g,Q'). \qquad (4)$$

Q' 是任意分法而与 Q 无关．但是对任意分法，

$$U(f+g,Q') \geqslant L(f+g,Q).$$

固定 Q'，对 Q 取上确界，然后再对 Q' 取下确界，即有

$$\inf_{Q'} U(f+g,Q') \geqslant \sup_{Q} L(f+g,Q).$$

联合(3)，(4)即有

$$\sup_{P} L(f,P) + \sup_{P} L(g,Q) \leqslant \sup_{Q} L(f+g,Q)$$
$$\leqslant \inf_{Q'} U(f+g,Q') \leqslant \inf_{Q'} U(f,Q') + \inf_{Q'} U(g,Q').$$

由于题设 f 与 g 可积，所以上式两端是相等的，其中间两项自然也相等．

(c)请读者注意，在本书中积分是用上、下和的确界来定义的．尽管在比较认真的微积分教材中都是这样讲的，但是人们总有这样一个想法：积分是积分和的“极限”．其实，积分和并不能排成序列，说积分是它的“极限”，那么是什么意义下的“极限”呢？所以许多书上说 $\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(z_i)(x_i - x_{i-1})$ 时，总要加上一句：“对一切分法，令 $\max|x_i - x_{i-1}| \to 0$”，“对 z_i 的一切取法”，不管做出什么样的“序列”，其“极限”都是一样的，然后定义这个“一样的”极限值为 $\int_a^b f(x)\,\mathrm{d}x$．而上面这一段话是很难懂的．因此，本书非常明确地就以确界概念为积分的基础，而完全不讲“极限”二字．这是很值得大家仔细思考的．

但是这样一来，一些原来大家以为很简单的事就难证了．习题3-3到习题3-6正是要求我们以确界概念为基础，来重新证

明积分的基本性质．以确界概念为基础就必然产生一个困难：即确界运算不是线性的．具体说，我们只能得到两个不等式

$$\inf(f+g) \geqslant \inf f + \inf g,$$
$$\sup(f+g) \leqslant \sup f + \sup g.$$

而不能写成等号．上面的解答提示就是告诉我们，如何克服由此带来的困难．

(c)也是一样，注意：

$$\inf(cf) = c\inf(f), \qquad \sup(cf) = c\sup(f)$$

是否正确，要看 c 的符号．如果 $c \geqslant 0$，它们是对的．如果 $c < 0$，请读者考虑应如何处理．

3-4 这个题目的困难却有点不同．如果 A 是一个 1 维的闭矩形（即闭区间），如图 A-8，则它的分法很简单，只要一些分点就可以实现．多加几个分点就实现了加细．但是本书是把"单积分"与"重积分"混在一起讲的，这时，分法与加细都比较复杂．即以本题为例，如果限于(x,y)平面情况，而且 A 是一个闭矩形(其实闭与不闭均可)，它的分法 P（图上的虚线）就要依靠两族直线(一族平行于 x 轴:$x=\alpha^i, i=1,2,\cdots,k$，另一族平行于 y 轴:$y=\beta^j, j=1,2,\cdots,l$)来实现．现在放一个 S 在 A 中(图上的粗线)，则需要把 P 加细为 P'，这时又要在每一族直线中各加一些 $x=\alpha^{i'}, y=\beta^{j'}$（图上的黑线)才能实现．如果从几何上已经看明白了平面(即 $\mathbf{R}^2$)的情况，而且知道如何构成分法，则对一般的 n 维情况也就明白怎样来做分法 P 并对之加细了(但具体计算仍然很繁冗)．

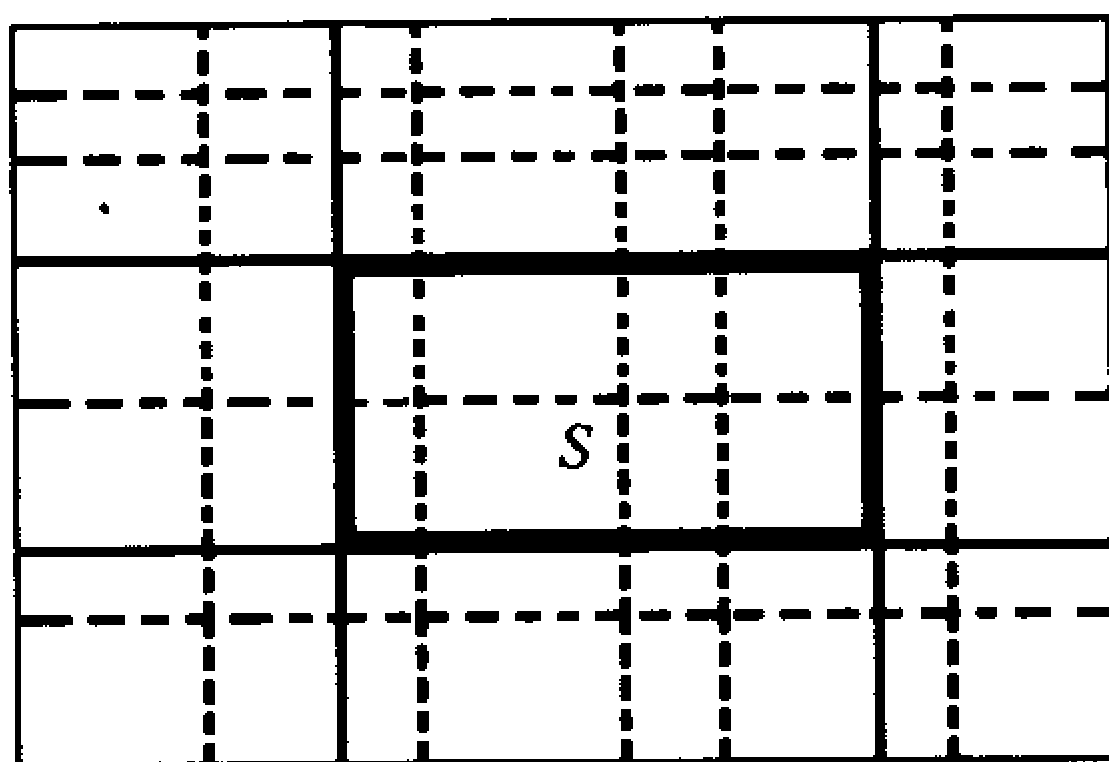


图 A-8

于是设已有 A 的一个分法 P，因为有了 $S \subset A$，所以把 P 加细为 P'. P' 中的子矩形可以分成两组，一组全在 S 内，一组则在 S 外. 于是 P' 就给出了 S 的一个分法 P'_S 和 $A-S$ 的分法 P'_{A-S}. 而下和 $L(f,P)$ 现在诱导出了 $L(f,P')$，后者又分成 $L(f,P'_S)$ 与 $L(f,P'_{A-S})$. 至此为止，我们有

$$L(f,P) \leqslant L(f,P') = L(f,P'_S) + L(f,P'_{A-S}). \qquad (1)$$

对于上和则有

$$U(f,P) \geqslant U(f,P') = U(f,P'_S) + U(f,P'_{A-S}). \qquad (2)$$

至此为止，除了加细引起了不等式外，其余都是等号.

把二式相减，有

$$\begin{aligned}[U(f,P'_S) - L(f,P'_S)] + [U(f,P'_{A-S}) - L(f,P'_{A-S})]\\ \leqslant U(f,P) - L(f,P).\end{aligned} \qquad (3)$$

而且（3）式左方两个方括号都是非负的.

由题设 f 在 A 上可积，故对任一 $\varepsilon > 0$ 必有一个分法 P 使（3）式右方 $< \varepsilon$（定理 3-3 的必要性部分）. 因此由 P 生成的 S 与 $A-S$ 的两个分法 P'_S 与 P'_{A-S} 均适合

$$U(f,P'_S) - L(f,P'_S) < \varepsilon,$$
$$U(f,P'_{A-S}) - L(f,P'_{A-S}) < \varepsilon.$$

再利用定理 3-3 的充分性部分知 $f|S$ 与 $f|A-S$ 均可积，而且易见

$$\int_A f = \int_S f|S + \int_{A-S} f|A-S.$$

如果 A 是有限个子矩形 S，T，⋯ 之并，对 $A-S$ 再利用以上方法即得

$$\int_A f = \sum_S \int_S f|S.$$

其实这只是我们熟知的公式 $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ 之推广．但是这里出现了新问题：本题中 A 与 S 都是矩形（开或闭），是特别简单的．如果它们都是形状极复杂的集合，甚至 A 是无穷多个这类集合之并，本题结论是否仍正确乃是复杂的问题．下面一节可积函数可以部分地回答它．

3-5 考虑 $g-f$，先证若 g 可积则 $-g$ 也可积（用3-3(c)），再证非负可积函数之积分也非负，最后用3-3（b）．

3-6 令 $f_+ = \begin{cases} f & f \geqslant 0 \\ 0 & f < 0 \end{cases}$，$f_- = \begin{cases} 0 & f \geqslant 0 \\ -f & f < 0 \end{cases}$．先证 $f_\pm$ 均可积再利用以上各题．

3-7 本题是来自对一个类似于狄利克雷函数的黎曼函数

$$f(x) = \begin{cases} 0 & x\text{ 是无理数}, \\ 1/q & x = p/q\text{ 是一个即约分数}. \end{cases} \quad x \in [0,1]$$

的研究．它与狄利克雷函数不同，是黎曼可积的．它的不连续点只是非0的有理数 x，有可数多个，因而与它黎曼可积不矛盾．本题的解法实际上就是通常研究黎曼函数的方法．

如图A-9，做任一分法 P，因为无理数的 x 在 $[0,1]$ 中稠密，所以在 P 之任一子矩形中必有这样的 (x,y)，其中 x 是无理数，因此 $f(x,y)=0$．但从定义来看，$f(x,y)\geqslant 0$，故 f 在任一子矩形 S 中下确界为0：$m_S(f)=0$．这样 $\sum\limits_S m_S(f)\,v(s)=0$，所以 $\sup\limits_P\{L(f,P)\}=0$．因此，现在只需证 $\inf\limits_P\{U(f,P)\}=0$ 即可．而为此，只需找一个分法 P 使 $0\leqslant \sum\limits_S M_S(f)\,v(S)<\varepsilon$（$\varepsilon$ 是任意正数）即可．我们来看一个最简单的分法，即把 $[0,1]\times[0,1]$ 用平行于两轴的直线等分为 N^2 个小正方形（N 待定）．注意到 $\sup f$ 其实只与 x 有关，所以我们只看同一个纵列中的许多小正方形．如果在某一纵列中含有一条直线 $x=p/q$（q 也待定），

而不含任何的 $x = p'/q'$, $q' < q$ 的直线. 则在这个纵列的各个小正方形中均有 $\sup_S f = 1/q$. 但有一件事要注意, $x = p/q$ 可能就是 P 的分线(图中的虚线). 这里 $\sup_S f = 1/q$ 就同时适用于这分线两侧的紧邻一列, 即在两列上同时有 $\sup_S = 1/q$.

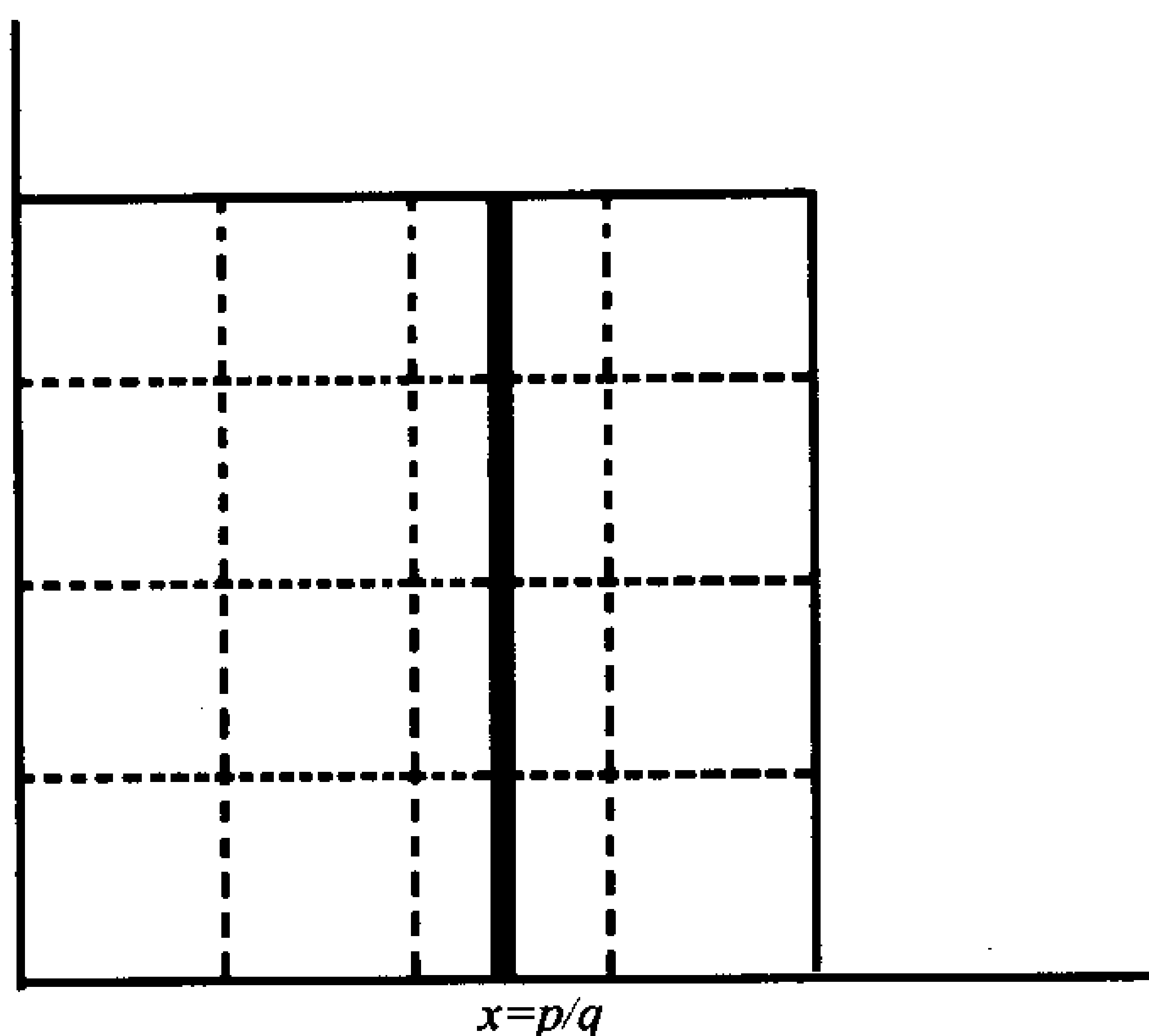


图 A-9

上面说这一纵列中不含有这样的直线 $x = p'/q'$, $q' < q$, 这怎么可能? 因为本题中所有分数 p'/q', p/q 均指既约分数, 故当 $q' = 1$ 时只有一条这样的直线 $x = 1/1$. $q' = 2$ 时有两条 $x = \frac{1}{2}$, $x = \frac{2}{2}, \cdots, q' = q - 1$ 时有 $q - 1$ 条. 总共有

$$Q = 1 + 2 + \cdots + (q - 1) = \frac{1}{2}q\ (q - 1)$$

条. 在这 Q 条直线上, $f(x,y) = 1/q' > 1/q$, 因此在含有它们(或以它们为边)的纵列上, $\sup_S f(x,y) > 1/q$. 但不论如何, $\sup_S f(x,y) \leqslant 1$. 这样的纵列最多有 $2Q$ 个而在其余的 $N - 2Q$ 个纵列上, $\sup_S f(x,y) \leqslant 1/q$ (注意 $\leqslant$ 不能误为 $=$, 因为有些纵列中含有 $x = p''/q''$), $q'' > q$, 而 $\sup_S f(x,y) = 1/q'' < 1/q$). 这样在计

算 $U(f,p)$ 时，应把各项分为两类，而有

$$U(f,p) = \sum_{\text{第一类}}\left(1 \cdot \frac{1}{N}\right) \sup_S f + \sum_{\text{第二类}}\left(1 \cdot \frac{1}{N}\right) \sup_S f$$

$$\leqslant 1 \cdot \frac{q(q-1)}{N} + \frac{1}{q} \cdot \frac{N - q(q-1)}{N} < \frac{q(q-1)}{N} + \frac{1}{q}.$$

对任意给定的 $\varepsilon > 0$，先定 q，使 $1/q < \varepsilon/2$，固定 q 后再令 N 充分大使 $\frac{q(q-1)}{N} < \frac{\varepsilon}{2}$. 合并起来即有

$$0 < U(f,P) < \varepsilon.$$

由 ε 之任意性，立即有

$$\inf_P \mathrm{D}(f,P) = 0,$$

原题得证.

3-8 设有有限多个闭矩形 $\{U_1,\cdots,U_h\}$ 可以覆盖这个闭矩形 $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$. 令 $V_i = U_i \cap I$，则因两个闭矩形之交仍为闭矩形（除非它是空集），所以闭矩形 $\{V_i,\cdots,V_k\}$ 仍可覆盖 I. 但是因 $V_i \subset U_i$，所以 $v(V_i) \leqslant v(U)$. 另一方面

$$\sum_{i=1}^{h} v(V_i) \geqslant v\left(\bigcup_{i=1}^{h} V_i\right) = v(I) = (b_1 - a_1)\cdots(b_n - a_n).$$

由假设，每一个因子 $b_j - a_j > 0$，所以上式右方为正，而

$$\sum_{i=1}^{h} v(U_i) \geqslant \sum_{i=1}^{h} v(V_i) \geqslant (b_1 - a_1)\cdots(b_n - a_n),$$

而不可能任意小. 因此 I 不可能为容度 0，

在这个证明中我们利用了闭矩形体积的“次可加性”

$$\sum_{i=1}^{h} v(V_i) \geqslant v\left(\bigcup_{i=1}^{h} V_i\right).$$

它在直观上是显然的.

3-9 (a)设集 A 有容度 0，则必有有限多个闭矩形 $\{U_1,\cdots,U_h\}$ 使 $A\subset\bigcup_{i=1}^{h}U_i$. 但是 $\bigcup_{i=1}^{h}U_i$ 在任一个坐标轴上的投影必定含于有限个闭区间之并集中，因此含于一个闭区集间中．$\bigcup_{i=1}^{h}U_i$ 则含于这些闭区间的乘积中．这个乘积可写为 $[a_1,b_1]\times\cdots\times[a_n,b_n]$，因此是有界集．$A$ 做为其子集当然也是有界的．总之容度 0 的集合必有界，而反过来无界集不可能有容度 0.

注意，我们并不需要 A 有容度 0，而只利用了 A 可被有限多个闭矩形覆盖．

(b)由(a)可知，我们只要做出一个无界的测度为 0 的集合即可．令 $\mathbf{Z}=\{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$ 为整数集，请读者证明它有测度 0. 为此选一个小区间包含 $k\in\mathbf{Z}$，其长为 l_k. 并设法令 $\sum l_k<\varepsilon$(ε 是任意小正数) 即可．这是容易的．另外要证明 $\mathbf{Z}$ 是闭集．读者可能从微积分教本中学过，如果一个集合包含其所有极限点则必为闭集．但是 $\mathbf{Z}$ 的极限点怎样找呢？所以我们应该回到定义：$A\subset\mathbf{R}^1$ 为闭集之定义即 $\mathbf{R}^1-A$ 为开．现在 $\mathbf{R}^1-\mathbf{Z}$ 是可数多个开区间 $(k,k+1)$，$k=0,\pm1,\pm2,\cdots$ 之并，因此为开．

3-10 (b) $C=\{(0,1)$ 中的有理点$\}$ 即可．请证明 C 有测度 0，再考虑 C 的边界是什么．

3-14 本章讲的可积函数都是有界的，因此 $f\cdot g$ 也是有界的．3-8 定理告诉我们，$f\cdot g$ 可积的必要条件是 $f\cdot g$ 的不连续点集合是测度 0 的．但是 $\{x:f\cdot g$ 在 x 不连续$\}\subset\{x:f$ 在 x 不连续$\}\cup\{x:g$ 在 x 不连续$\}$. 由此证明右式为测度 0 集合．

(请读者考虑，上式的“$\subset$”能不能换成“$=$”？为什么？)

3-16 不妨设 $A\subset\mathbf{R}$ 为区间 $[0,1]$，$C=\{x:x$ 为有理数$\}$. 先证 C 为有界且为测度 0. 但是 χ_C 作为 A 上的函数处处不连续，因此由 3-8 定理即得．

3-18 若$f:A\to\mathbf{R}$是连续的非负函数且$\int_A f=0$，证明$f\equiv 0$是通常微积分教本中的习题，但若去掉f为连续这个条件，证明就困难多了.

固定一个m，并设$\{x: f>\frac{1}{m}\}$不是容度0. 为简单计，设A是$\mathbf{R}^1$中的闭区间$[0,1]$（本节中的A是$\mathbf{R}^n$中的闭矩形）. 把A等分为$[\frac{k}{N},\frac{k+1}{N}]$之并：$A=\bigcup_{k=0}^{N-1}[\frac{k}{N},\frac{k+1}{N}]$，则$\{x: f>\frac{1}{m}\}$成为$N$个集合$\{x:\frac{k}{N}\leqslant x\leqslant\frac{k+1}{N}\}\cap\{x: f>\frac{1}{m}\}$之并．这$N$个集合中必至少有一个不是容度0. 把这个子集合再$N$等分，仿此以往，就会得到一个矛盾.

3-23 本节中的记号是指$\mathfrak{L}=\mathfrak{L}(x)=\mathbf{L}\int_B f(x,y)\mathrm{d}y$，$\mathscr{U}=\mathscr{U}(x)=\mathbf{U}\int f(x,y)\mathrm{d}y$. 由定理3-10，$\mathscr{U}$与$\mathfrak{L}$均为$A$上的可积函数而且$\int_{A\times B}\chi_C=\int_A\mathscr{U}(x)$. 注意在一个集合“有容度0”的定义中并未用“容度”的概念，而现在又定义了$\int_C 1=\int_{A\times B}\chi_C$为$C$之“容度”，那么要问$C$“有容度0”与先计算出$C$之“容度”后来又发现这个“容度”为0是不是一回事？答案是肯定的，但可惜的是书上没有讲．如果我们承认了这一点，则本题十分容易．事实上由于已设C有容度0，所以$\int_{A\times B}\chi_C=0$，亦即$\int_A\mathscr{U}(x)=0$. 那么要问$\mathscr{U}(x)$在本题中是什么．它是$\mathscr{U}(x)=\int_{固定x}\chi_C(x,y)\,\mathrm{d}y$即是集合$C$与直线$x=$常数的交集：$C\cap\{(x,y):x固定\}$（不妨记为$C(x)$）之特征函数$\chi_{C(x)}$的积分．所以就是$C(x)$之容度．显然$\mathscr{U}(x)\geqslant 0$，而由$\int_A\mathscr{U}(x)=0$即有$\{x:C(x)之容度\neq 0\}$

具有测度0（习题3-18）．这就是本题所求证的．

现在我们来对χ_C证明上述两个概念是相同的．若C“有容度0”，则必有有限多个闭矩形$R_1,\cdots,R_h$使$C\subset\bigcup_{j=1}^{h}R_j$，而且$\sum_j v(R_j)<\varepsilon$，$\varepsilon$是任意小正数．但是很容易$\sum_{j=1}^{h}v(R_j)$不小于$C$的一个上和，从而有一个上和$<\varepsilon$，而$\chi_C$之上积分为0. χ_C之下积分自然为0，从而$\int\chi_C=0$. 反过来，若$\int\chi_C=0$，则对任一正数ε，χ_C必有一个上和$<\varepsilon$. 就是说C必有一个分法$P:C=R_1\cup\cdots\cup R_h$（$R_j$为闭矩形）适合$\sum_{j=1}^{h}\sup_{R_j}\chi_C\cdot v(R_j)<\varepsilon$. 但$\sup_{R_j}\chi_C=1$，所以$\sum_{j=1}^{h}V(R_j)<\varepsilon$，亦即$C$“有容度0”．

本题可能有其他证法，但实质上都和上述相同．

3-25　当$n=1$时结论恒成立．设对$n-1$维情况已证明了这个结论．现在对n维情况来证明它．记$C_n=[a_1,b_1]\times\cdots\times[a_n,b_n]$，如果$C_n$有测度0，由习题3-17有$\int\chi_{C(n)}=0$. 但是

$$\int\chi_{C(n)}=\int_{C(n-1)}\chi_{[a_n,b_n]},$$（请读者说明此式的意义．）

由习题3-17又有$\chi_{[a_n,b_n]}=0$.（这个说法不严格，请读者补正．）但这是不对的，因为$\chi_{[a_n,b_n]}=b_n-a_n>0$.

3-27　作一个区域$C=\{(x,y):a\leqslant x\leqslant b,x\leqslant y\leqslant b\}$. 它是一个三角形如图A-10. 请读者自己证明$\chi_C$是可积函数．从而$\chi_C\cdot f(x,y)$也可积（习题3-14）．对每个固定的$x$，证明$\chi_C\cdot f(x.y)$是$y$的可积函数，再用定理3-10之注2即有

$$\begin{aligned}\int_{[a,b]\times[a,b]}\chi_c f(x,y)\mathrm{d}x\mathrm{d}y&=\int_a^b\mathrm{d}x\int_x^b\chi_C f(x,y)\mathrm{d}y\\&=\int_a^b\mathrm{d}x\int_x^b f(x,y)\mathrm{d}y.\end{aligned}$$

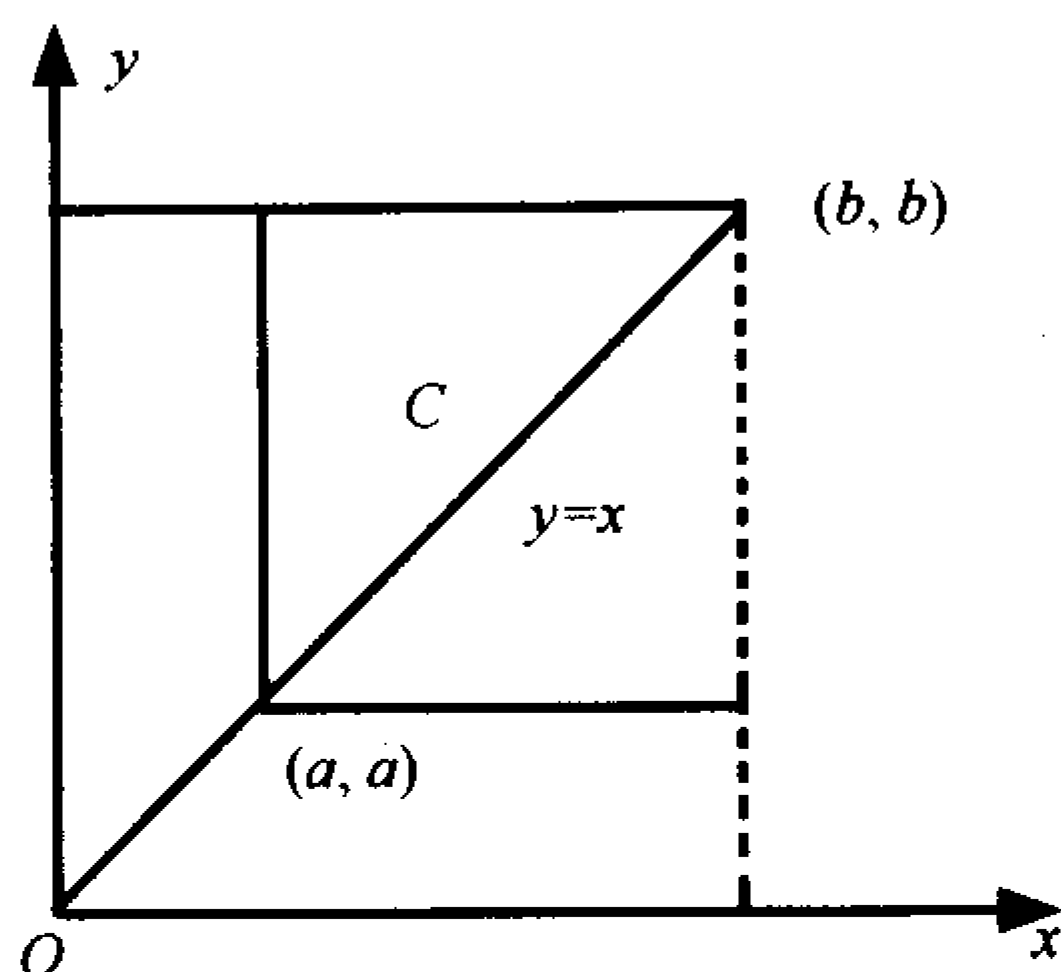


图 A-10

交换积分次序即得.

3-29 改变一下记号，把 y 记成 r，而且设这个集 C 全位于 $r>0$ 处. 如果记 C 绕 z 轴旋转 θ 后所得的集为 $C(\theta)$，则 C 与 $C(\theta)$ 同为约当可测，而且其面积一样. C 旋转一周后成一个旋转体，用富比尼定理计算其体积.

如果 C 全位于 $y>0$ 处，这个假设不成立，本题的结论应如何改变?

3-30 习题 1-17 中的集 A 有多种多样做法. 但是不管是何种做法，这样做出的 A (即题中之 C) 边界总是 $[0,1]\times[0,1]$. 这个边界有零测度吗? 这时 χ_C 可积吗（定理 3-9)?

注意，定理 3-10 之注 2 表明，如果 $\int\limits_{A\times B} f$ 在 $A\times B$ 上可积（即 $\int\limits_{A\times B} f$ 存在)，而且 $g_x(y)=\int\limits_B f(x,y)\mathrm{d}y$ 在 A 上可积, $h_y(x)=\int\limits_A f(x,y)\mathrm{d}x$ 在 B 上也可积. 则

$$\int\limits_{A\times B} f=\int\limits_A \mathrm{d}x\int\limits_B f(x,y)\mathrm{d}y=\int\limits_B \mathrm{d}y\int\limits_A f(x,y)\mathrm{d}x.$$

但是这个题目告诉我们，哪怕后两个积分（逐次积分）存在且相等，第一个积分（重积分）也不一定存在. 产生这样的问题在于我们采用的积分定义有待改进.

3-33 本题最好再加上一个条件：$f(x,y)$ 本身也在$[a,b]\times[a,b]$上连续．以一个变量的情况为例，因为如果只有$f(t)$ 可积而没有连续性，$\int_a^x f(t)\,dt$ 如何对 x 求导是一个很复杂的问题．如果读者在经典的微积分中学过相关的定理．就可以不必加上条件来做本题．

3-35 这是一个线性代数的题目．对本章讲到的变量变换以及下面两章都起很关键的作用．因此，读者应该努力熟悉它所包含的思路和技巧．由于本章的规定，$\{e_1,\cdots,e_n\}$ 表示 $\mathbf{R}^n$ 中标准正交基底，就是说，它们是 n 个线性无关的向量，而且 $\langle e_i,e_j\rangle=\delta_{ij}$. 于是 $\mathbf{R}^n$ 中每一个向量 x 都可以表示为 $x=x_1e_1+\cdots+x_ne_n$，而我们可以用（$x_1,\cdots,x_n$）表示这个向量．现在 $x_1,\cdots,x_n$ 都是实数，与 $e_i,\cdots,e_n$ 为向量不同，这些实数称为 x 对此基底的坐标．例如

$$e_i=1\cdot e_1+0\cdot e_2+\cdots+0\cdot e_n,$$

所以 e_1 的坐标表示是$(1,0,\cdots,0)$，这就是本书第 1 章的讲法．不过，为方便起见，我们采用竖（列）向量的写法：

$$x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}.$$

(a)重要的是把 g 用矩阵表示出来，例如第一个，有

$$g(x)=g\left(\sum_{i=1}^n x_ie_i\right)=\sum_{i=1}^n x_ig(e_i)=\sum_{i\neq j}x_ie_i+ax_je_j.$$

亦即

$$g\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}x_1\\\vdots\\ax_j\\\vdots\\x_n\end{pmatrix}\quad\text{--- 第}j\text{个元}\qquad(2)$$

它的几何意义如下：如果以 $e_1,\cdots,e_n$ 为稜做出一个单位长方体 V，则在用 g 作用后，其第 j 个稜伸长了成 a 倍（请读者考虑 a 的符号对 g 的几何意义的影响），而其他的稜不变．因此 $g(V)$ 仍是一个长方体，其体积为 $|a|v(V)$．由(2)又得 g 的矩阵表示为

$$g = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \text{--- 第} j \text{行}$$

而

$$\det g = \begin{vmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{vmatrix} = a.$$

因此有

$$v(g(V)) = |a|v(V) = |\det g|v(V).$$

请读者同类似方法考虑后两个．第二个 g 在几何上表示“切变”，在切变下体积是不变的，这是一个非常简单的几何定理，如图 A-11．读者可以依照上一部分把这个情况详细写出，特别是写出 g 的矩阵表示．

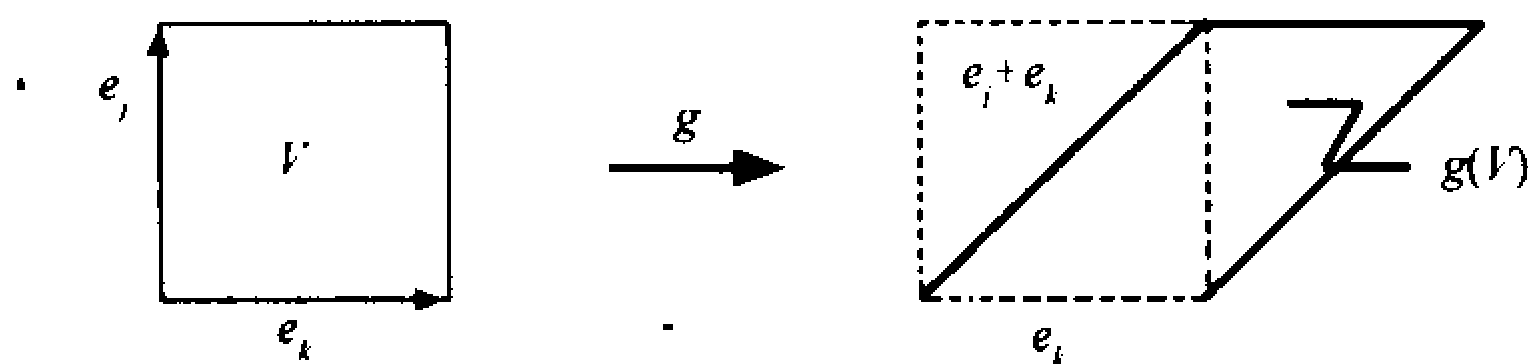


图 A-11

第三个情况是 e_i 与 e_j 对调．我们最熟悉的是 $\mathbf{R}^2$ 中 x 轴与 y 轴的对调．这时坐标系由右手系变成了左手系，称之为定向的改变．问题在于 V 之体积 $v(V)$ 有没有符号？在讲重积分时，小矩形的体积一定取非负的，但到了第 5 章，体积将是有符号的，而且若我们规定某一个定向(例如右手系下)的体积为非负，则定向改变后体积要变号．这样我们会有

$$v(g(V)) = -v(V).$$

总之有

$$v(g(V)) = \det gv(V).$$

但是按本章的提法，我们仍有

$$v(g(V)) = |\det g|v(V).$$

理解这个区别是理解第 4、5 两章的关键．也请读者写出这个情况下 g 的矩阵表示．

至于证明 $v(g(U)) = |\det g|v(U)$，要注意，至少从题面上看并没有设 U 的稜就是 $e_1,\cdots,e_n$ 方向的向量．因此应该用

$$\int\chi_U \quad 和 \quad \int\chi_{g(U)}$$

来计算体积，并且看它们之间的关系．而在例如计算 $\int\chi_U$ 时，我们要计算其上、下积分．如果作一个分划 P 把 U 划分为许多其稜平行于 $e_1,\cdots,e_n$ 的小矩形．任取一个小矩形 R，如果完全在 P 中，则 $\sup\limits_R \chi_U = \inf\limits_R \chi_U = 1$. 而若 R 有一部分在 P 外，则

$$\sup_R \chi_U = 1, \qquad \inf_R \chi_U = 0.$$

上、下积分之差别在此．读者应该自己证明 $\int\chi_U$ 即 U 之体积．至于 $g(U)$，则 g 把覆盖 U 的小矩形都作了变动．但这里有一个问题，即在"切变"下小矩形变成小的平行体．因此上面的讨论要改变，即分法需推广为用小平行体而不是小矩形来实现，

这不是困难的．又请读者把以上证明的不严格处补正．

4. 链上的积分

4-1 (a)请注意由定理4-4的(3)有

$$\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}=\frac{(1+1+\cdots+1)!}{1!1!\cdots1!}\mathrm{Alt}(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k})$$
$$=k!\,\mathrm{Alt}(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}). \tag{1}$$

再由交代算子 Alt 与张量积的定义，

$$\mathrm{Alt}(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k})(e_{i_1}\otimes\cdots\otimes e_{i_k})$$
$$=\frac{1}{k!}\sum_{\sigma}\mathrm{sgn}\,\sigma(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k})(e_{\sigma(i_1)}\otimes\cdots\otimes e_{\sigma(i_k)})$$
$$=\frac{1}{k!}\sum_{\sigma}\mathrm{sgn}\,\sigma\delta_{i_1\sigma(i_1)}\cdots\delta_{i_k\sigma(i_k)}=\frac{1}{k!} \tag{2}$$

合并 (1),(2) 即得原题之证．可见 $\wedge$ 的定义中如果没有因子 $\frac{(h+l+\cdots+m)!}{k!l!\cdots m!}$, (2) 中的因子 $\frac{1}{k!}$ 将难于清除，而会得到

$$\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}(e_{i_1},\cdots,e_{i_k})=\frac{1}{k!}.$$

这将是很不自然的．因为在 $\mathbf{R}^3$ 的情况，若 $k=2$，外积与向量积是十分相似的，而 $\varphi_1\wedge\varphi_2$ 是以 φ_1，φ_2 为稜的平行四边形的面积，前面加一个因子 $\frac{1}{k!}=\frac{1}{2!}$ 将得到由两个向量 φ_1，φ_2 所组成的三角形的面积．在对于一般的 k, 我们将得到一个 k 维平行体的有符号的体积 $\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}$, 而 $\frac{1}{k!}\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}$ 是一个 k 维“单形”的有符号的体积．本书没有介绍单形，而是以 k 维正方体为基础．后者是区间、正方形、立方体……的推广，而前者则是区间、三角形、四面体……的推广．

(b)如果我们将 $v_1,\cdots,v_k$ 用基底 $e_1,\cdots,e_n$ 来表示，而得

$$v_i = \sum_{j=1}^{n} a_{ij}e_j, i = 1,2,\cdots,k. \tag{3}$$

则由(a)有

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1,\cdots,v_k) = \sum_j \sum_\sigma \operatorname{sgn} \sigma a_{1i_1}\cdots a_{ki_k} \tag{4}$$

这里的 σ 是由 $i_1,\cdots,i_k$ 所成的排列. 把(3)写成矩形

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} a_{11}\cdots a_{1n} \\ \cdots \\ a_{k1}\cdots a_{kn} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

注意这里竖写的并不是一个向量，而是把 k 个(或 n 个) 向量竖写，再“借用”矩阵乘法的“规则”，所以只是一个方便的记法. 而(4)式显然是这个矩阵的第 $i_1,\cdots,i_k$ 列（依此次序）所成的 k 阶行列式.

4-2 f^* 是一个线性变换，注意到 $\dim\Omega^k$ 的维数是 $\binom{n}{k}$. 所以 Ω^n 的维数为1. 1维空间上的线性变换只能是乘以常数 c. $c=\det f$ 的证明只要把 f 写出来就容易看见了.

4-3 按原书提示的矩阵写法，$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = A\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$，先计算 $A \cdot A^{\mathrm{T}}$.

4-5 det 是一个 n 阶交代张量，所以是对每一个 v 都连续，c 也是连续的，因此 $\det \circ c$ 也是连续的. 但是基底只有两类，它的“连续”变化就只能是始终保持在同一类中.

4-6 (a) 叉乘“×”是 $\mathbf{R}^n$ 中的 $n-1$ 个向量 $v_1,\cdots,v_{n-1}$ 到 $\mathbf{R}^n$ 中的一个重线性的“交代的”映射. 当 $n=2$ 时，就应是 $\mathbf{R}^2$ 中一个向

量 v（即 v_1）到另一个向量 z（记作 $z = v\times$）的映射．而且使 $<\omega,z> = \det\begin{pmatrix} v \\ \omega \end{pmatrix}$ 对一切 $\omega \in \mathbf{R}^2$ 成立．我们现在来给出 $z = v\times$ 的具体表达式．如果对 $\mathbf{R}^2$ 的一个标准正交基底 $\{e_1, e_2\}$，我们有

$$v = ae_1 + be_2 \text{ 或记作 } v = (a,b),$$
$$w = xe_1 + ye_2 \text{ 或记作 } w = (x,y),$$
$$z = Ae_1 + Be_2 \text{ 或记作 } z = (A,B).$$

则因 $<w,z> = Ax + By$, $\det\begin{pmatrix} v \\ w \end{pmatrix} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx$，而且此式需对一切 (x,y) 均成立，故有

$$A = -b.\ B = a.$$

可知在 $n=2$ 情况下"$\times$"这个映射就是

$$\times : v = (a,\ b) \mapsto z = v\times = (-b, a). \tag{1}$$

这里当然没有"重"线性，而只有线性，也谈不上"交代"．而且从几何上看，(1)就是旋转$\frac{\pi}{2}$.

(b)对一般的 n，情况当然没有这么简单，这时我们有

$$<w,z> = <w, v_1 \times \cdots \times v_{n-1}> = \det\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix} \tag{2}$$

对一切 $w \in \mathbf{R}^n$ 成立．和上面一样，令

$$v_i = \sum_{j=1}^{n} a_{ij} e_j, \qquad i = 1,2,\cdots,n-1.$$
$$w = \sum_{j} x_j e_j,$$
$$z = v_1 \times \cdots \times v_{n-1} = \sum_{j=1}^{n} \alpha_j e_j.$$

代入(2)式双方，即有

$$\sum_{i=1}^{n}\alpha_i x_i = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{n-1\,1} & \cdots & a_{n-1\,n} \\ x_1 & \cdots & x_n \end{vmatrix}$$

把右方行列式按最后一行展开，并记 x_i 的代数余子式为 A_i，即有 $\alpha_i = A_i$.

特别是，若令 $w = z = v_1 \times \cdots \times v_{n-1}$ 即有

$$\det\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_1 \times \cdots \times v_{n-1} \end{pmatrix} = \begin{vmatrix} a_{11} & \cdots & \alpha_{1n} \\ \vdots & \cdots & \vdots \\ \alpha_{n-1\,1} & \cdots & a_{n-1\,n} \\ \alpha_1 & \cdots & \alpha_n \end{vmatrix} = \sum_{i=1}^{n} A_i^2 > 0.$$

右方必为正而不仅是≥0，这是因为已设 $v_1,\cdots,v_{n-1}$ 线性无关，所以至少有一个 $A_i \neq 0$.

因为这个行列式为正，所以 $[v_1,\cdots,v_{n-1},v_1 \times \cdots \times v_{n-1}]$ 是通常的定向.

4-9 本题是我们熟知的向量的“向量积”（本书称为叉积）的定义和一些基本性质的新讲法. 在我们熟知的讲法中，向量 $\vec{A}$ 和 $\vec{B}$ 的向量积 $\vec{A} \times \vec{B}$ 定义为一向量 $\vec{C}$. $\vec{C}$ 与 $\vec{A}$，$\vec{B}$ 均正交，大小为 $|\vec{A}|\cdot|\vec{B}|\sin(\vec{A},\vec{B})$（即 $\vec{A}$，$\vec{B}$ 所成的平行四边形的面积)，而 $\vec{C}$ 之指向应使 $\vec{A},\vec{B},\vec{C}$ 构成一个右手坐标系. 这里是假设 $\mathbf{R}^3$ 中已经有了一个标准的正交坐标系 O-xyz，而且是右手系. 本书中的讲法则不同. 它同样设 $\mathbf{R}^n$ 中有通常的标准正交基底 $\{e_1,\cdots,e_n\}$（相当于 O-xyz 轴)，而且有一个取定了的定向 $\mu = [e_1,\cdots,e_n]$，这自然相当于右手系的规定. 但在高维空间中无所谓左、右手，所以在两种可能的定向中要事先选定一种. 特别重要的是，本书

在 $\mathbf{R}^n$ 中只承认 $n-1$ 个向量的叉积 $v_1\times\cdots\times v_{n-1}$，而不承认少于 $n-1$ 向量的叉积，而我们熟知的向量计算只讲两个向量的向量积 $\vec{A}\times\vec{B}$，所以只有 $n-1=2$ 即 $n=3$ 时，这两种讲法才有可能统一．我们熟知的教材中讲到向量积时，一般地不强调它只在 $\mathbf{R}^3$ 中才有意义，这一点极易引起误会．例如，若 $\vec{A},\vec{B}$ 是 xy 平面上的向量又如何？这时 $\vec{A}\times\vec{B}$ 既然在 z 轴方向上，则已越出 $\mathbf{R}^2$ 即 (x,y) 平面，所以"$\times$"不是 $\mathbf{R}^2$ 中的向量运算．

用本书的讲法，是讲 $n-1$ 个向量的叉积．那么能否讲 2 个向量的叉积呢？是可以的，但其结果是一个 2 阶交代张量，而不是向量．k(小于 $n-1$) 个向量之叉积是一个 k 阶交代张量．那么 $n-1$ 个向量的叉积量不应该是一个 $n-1$ 阶交代张量吗？答案是肯定的．但是这种张量构成 n 维空间，且与 $\mathbf{R}^n$ 同构．同构的东西可以看成同样的东西，所以本书说 $v_1\times\cdots\times v_{n-1}$ 是一个向量．这类问题在重线性代数理论中有完美的叙述．

本题的目的就是要读者利用 $v_1\times\cdots\times v_{n-1}=z\in\mathbf{R}^n$，而 $\mathbf{R}^n$ 中的向量 z 又由 $<w,z>$ 来定义：

$$<w,z>=<w,v_1\times\cdots\times v_{n-1}>=\det\begin{pmatrix}v_1\\ \vdots\\ v_{n-1}\\ w\end{pmatrix}. \tag{1}$$

利用它来在 $\mathbf{R}^3$ 中重新证明我们熟知的向量积的性质．

(a) 请看 $\det\begin{pmatrix}e_1\\ e_1\\ w\end{pmatrix}=?$ $\det\begin{pmatrix}e_1\\ e_2\\ w\end{pmatrix}=?$ $<w,e_3>=?$注意到 $e_1=(1,0,0)$，$e_2=(0,1,0)$，$e_3=(0,0,1)$ 即得．

(b) 先证明叉积的分配律：

$$< w,(u+u')\times v > = \det\begin{pmatrix} u+u' \\ v \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \det\begin{pmatrix} u' \\ v \\ w \end{pmatrix}$$

$$= < w,u\times v > + < w,u'\times v >,$$

$$(u+u')\times v = u\times v + u'\times v,$$

再利用(a).

(c)与(e)是一组，不妨先证(e). 再注意到 $< v\cdot w > = |v|\ |w|$ $\cos\theta$ 即可得(c).(e)本身的证明是一个著名的恒等式:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\leqslant k<j\leqslant n}(a_k b_j - a_j b_k)^2. \quad (2)$$

此式十分有用，而且不难证明.

(c)中后两个式子 $< v\times w,v > = < v\times w,w > = 0$ 就是 $\vec{A}\times\vec{B}$ 与 $\vec{A}$, $\vec{B}$ 都正交.

(d) 第一个式子是混合积，它说明混合积的几何意义就是以 v, w,z 为稜的平行体的有符号的体积.

另两个恒等式易证.

4-11 因为 $\{v_1,\cdots,v_n\}$ 是一个基底，所以 $f(v_i)$ 仍可用这个基底来表示:

$$f(v_i) = \sum_{j=1}^{n} a_{ij}v_j. \qquad i = 1,\cdots,n.$$

又因为这个基底是标准正交的，所以若求 v_k 与上式双方的内积(在 T 意义下的内积)，当有

$$T(f(v_i),v_k) = \sum_{j=1}^{n} a_{ij}T(v_j,v_k) = \sum_{j=1}^{n} a_{ij}\delta_{jk} = a_{ik}.$$

同样，考虑 $T(v_i, f(v_k))$ 知

$$T(v_i, f(v_k)) = a_{ki}.$$

由 T 之自伴性，$T(f(v_i),v_k) = T(v_i, f(v_k))$，而得 $a_{ik} = a_{ki}$.

本题告诉我们 $\mathbf{R}^n$ 到 $\mathbf{R}^n$ 的自伴算子在标准正交基底下就表示为 n 阶对称矩阵.

4-13 f_x 就是由 Df 所表示的线性映射，称为相应于 f 在 D 点的切映射. 它的几何意义由图 A-12 自明.

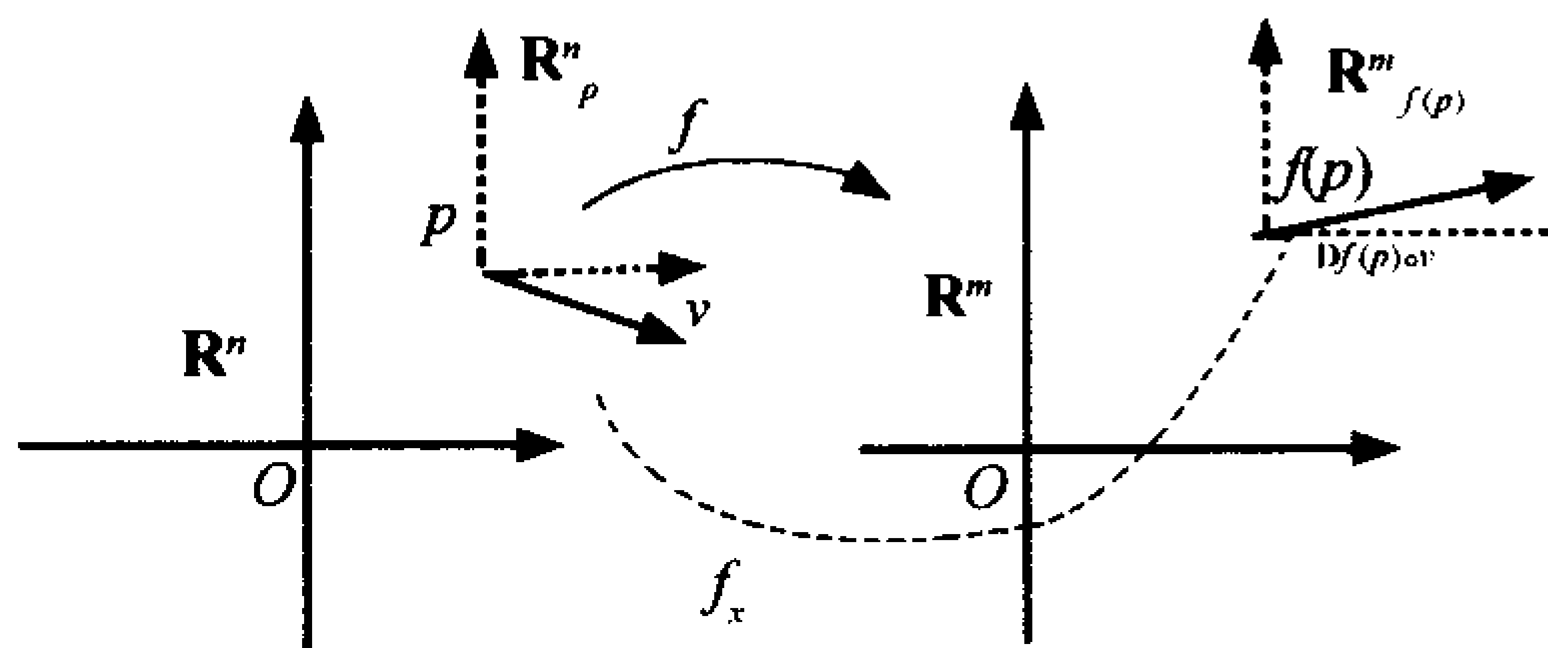


图 A-12

(a) 由此考查 $(g\circ f)_*$，对任一向量 $v \in \mathbf{R}^n_p$

$$(g\circ f)_*(v_p) = \mathrm{D}(g\circ f)(p)V_{g\circ f(p)}.$$

利用链法则

$$\mathrm{D}(g\circ f)_p = (\mathrm{D}g)_{f(p)}\cdot(\mathrm{D}f)_p.$$

代入上式即得原题之证：

$$(g\circ f)_*(v_p) = g_*|_{f(p)}\circ f_*|_p v(p).$$

可见 (a) 就是链法则.

(b) 是与 (a) 对偶的关系式，它的基本定义是

$(g\circ f)^x\omega(P)(v_1,\cdots,v_k) = \omega((g\circ f)(p))((g\circ f)_*v_1,\cdots,(g\circ f)_*v_k)$

$= \omega(g(Q))(g_*\circ(f_*v_1),\cdots,g_*\circ(f_*v_k))_{(g\circ f)(p)}$ 这里令 $Q = f(p)$

$= g^*\omega(g(Q))(f_*v_1,\cdots,f_*v_k)(p)$

$= (f^*\circ g*)\omega(g\circ f)(p)(v_1,\cdots,v_n)p.$

而原式得证.

本题的重要性在于，若由 f_* 过渡到对偶的 f^*，则映射复合的次序要反转.

4-14 把 c 看作由 $[0,1]$ 到 $\mathbf{R}^2$ 的映射（即变换），并考虑相应的 c_*. 为此先应考查 $\mathrm{D}\,c(t)$ 以及 $[0,1]$ 的单位切向量 e_1.

4-17 如果记 f 之定义域 $\mathbf{R}^p$ 中的点为 x，值域 $\mathbf{R}^p$ 中的点为 y，并将 f 写成 $\mathbf{R}^p_x \to \mathbf{R}^p_y$，而且 $y = f(x)$. 如果说 $\mathbf{R}^p_x$ 或 $\mathbf{R}^p_y$ 中的点 x 与 y 均为向量：

$$x = x_1 e_1 + \cdots + x_p e_p, \quad y = y_1 e_1 + \cdots + y_p e_p.$$

这样一来，原来我们是把 $f:\mathbf{R}^p \to \mathbf{R}^p$ 看成一组 p 个 p 元函数，现在则看成一个向量空间 $\mathbf{R}^p_x$ 到另一个向量空间的映射（即变换）. 例如 $x_1, x_2, \cdots, x_p$ 是看作 p 个互相独立的自变量，现在则看成一个 p 维向量的分量.

这里对原题的条件要加一些说明，首先 f 应该是可微的，否则切映射、散度等均不能定义. 其次 f 的定义域不一定是整个 $\mathbf{R}^p_z$，而可一般地取为一个开集 $D \subset \mathbf{R}^p_x$. 开集的条件是重要的，因为开集是不含边界点的，而一个函数（或变换）在边界点上“可微”是很难解释的. 同样，f 之值域也不一定是全空间 $\mathbf{R}^p_y$，甚至也不必是开集，我们就记之为 $f(D) \subset \mathbf{R}^p_y$ 好了. 于是原题变成“若 $f:D \subset \mathbf{R}^p_x \to F(D) \subset \mathbf{R}^p_y$ 是可微的，则……”.

本题的(a)就是让读者明白，一组 p 个 p 元函数就是 $D \subset \mathbf{R}^p_x$ 中的向量场.

把这个向量场用局部坐标写出来以后，(b)的证明自明.

4-18 前一部分的作法和上题相仿. 后一部分是十分重要的，我们给出它的几何思想，并请读者把它严格化（可略去高阶无穷小）. 作 f 的两个等值曲面 $f(p) = c_1$ 和 $f(p) = c_2$，如图 A-13. 过 P 作法线交 $f(p) = c_2$ 于 Q，则 f 在“自变量” P 点走过一个距离 PQ 时变化了 $\Delta f = c_2 - c_1$. 如果沿另一条路径走到 Q'，Δf 仍然是 $c_2 - c_1$. 可见只要 PQ 最短，f 就变化最快，而最短的路径显然是沿法线 PQ. 我们已知，$\mathrm{grad}\,f$ 的方向是等值面 $f(p)$ = 常数的法线

上 f 上升的方向，故原题得证.

如果希望把它严格化，则例如有

$$|\Delta f| = |c_2 - c_1| = |f(Q') - f(P)| \approx |(\mathrm{D}f)(P)v'| = |(\mathrm{D}f)(P)|\ |v'|$$

但是 $|v'|\cos\theta = |v|$，因此易得本题结果. 上面我们略去了些什么？为什么可以略去？

4-19 这个题目的意思是把我们在熟知的微积分教程中的一个"难点"——场论用微分形式来写. 这样即可得到十分简明而且容易记忆的讲法. (a)是说为何把 grad，curl，div 写成不同阶数的微分形式. (b)其实就是一个公式 $\mathrm{d}^2 = 0$（定理 4-10 之(3)）. (c)就是定理 4-11.

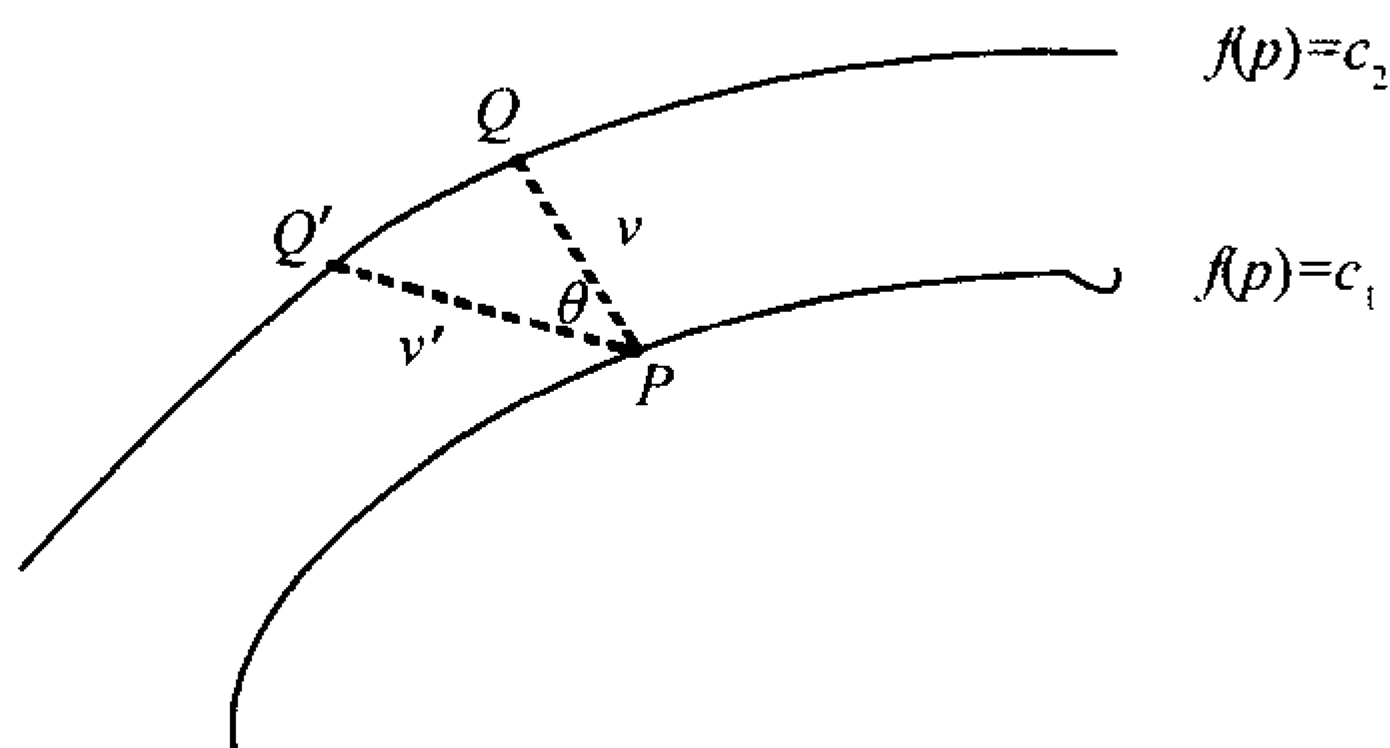


图 A-13

可是我们切勿以为这只是形式的类比. (a)告诉我们，grad，curl，div 分别在 $\Omega^1(\mathbf{R}^3_p)$，$\Omega^2(\mathbf{R}^3_p)$，$\Omega^3(\mathbf{R}^3_p)$ 中，因此可以自然地推广到 $\Omega^1(\mathbf{R}^3_p), \Omega^2(\mathbf{R}^3_p), \cdots, \Omega^n(\mathbf{R}^n_p)$ 中.（从 Ω^3 到 Ω^{n-1} 缺少了，但是用微分形式的理论同样可以处理，只不过没有那么明显的几何意义罢了. 从这个题目，也可以再回到前面讲叉积的说明. $\vec{A} \in \Omega^1(\mathbf{R}^3_p)$，$\vec{A} \times \vec{B} \in \Omega^2(\mathbf{R}^3_p)$（记号不严格），而在一般的 $\mathbf{R}^n$ 中则只能讨论 $v_1 \times \cdots \times v_{n-1}$ 了. 这正是重线性代数的很本质的地方.

我们还应特别注意（c）中要求 A 为星形集. 定理 4-11 中也是作此要求的. 其中最重要之点是可以在中心 O 与任一点的

连线上作积分．请读者再回到习题2-23，在那里我们特别强调了拉格朗日公式一定要用于定义在一区间上的函数．拉格朗日公式就是微分学的中值定理．我们还常用到积分学的中值定理，在那里也要用到“积分区域”是一个区间．

4-21 请参看习题3-41中如何定义θ，并请读者再回头想一下，何以θ的“定义”会如此复杂？本题讲到“在θ有定义的集合上”，这就是说，要除去点$(0,0)$．为什么？本题同样是很重要的．如果不细心，就会得出$\mathrm{d}\ \mathrm{arctg}\,\frac{y}{x}$．但是这种记法是有缺点的．因为显然题目已将$(x,y)=(0,0)$排除，因为这时$\theta$没有定义．但$x=0$时，$\theta$确有定义（例如可以是$\frac{\pi}{2}$）但$\mathrm{arctg}\,\frac{y}{x}$没有意义．这样就容易理解习题3-41了．

4-22 这是在数学中表示“有限和”的一个常用的方法．问题在于φ中的奇异n维立方体的数目（准确些说是φ的基数）“很大”，不但不一定是有限的，也不一定是可数无限的，甚至不一定能把φ中每个奇异n维立方体对应于$\mathbf{R}^1$的连续变量$\alpha:\alpha\in\mathbf{R}^1$，这时我们至少还可以说$\varphi$具有连续优势．于是我们就说，“可以定义一个函数$f:\varphi\to\mathbf{Z}$”使得只有有限多个$c\in\varphi$使$f(c)\neq0$，而其他的$c$必使$f(c)=0$．如果把使$f(c)\neq0$的$c$之集合记作$\mathbf{C}$，则$\mathbf{C}$是一个有限集．不失一般性可以设$\mathbf{C}=\{1,2,\cdots,k\}$．相应地$c\in\varphi$也记为$c_1,\cdots,c_k$．这样可用记号

$$f=\sum_{i=1}^{k}f(c_i)c_i=\sum_{i=1}^{k}a_ic_i,\qquad a_i=f(c_i)\in\mathbf{Z}$$

表示f，而f就成了形式的有限和，我们称之为n维链．如果有两个n维链如$f=\sum\limits_{i=1}^{k}a_ic_i$，$g=\sum\limits_{j=1}^{l}b_jd_j$，则不但不一定有$a_i$等于某个$b_j$，甚至$\{c_1,\cdots,c_k\}$与$\{d_1,\cdots,d_l\}$可以根本不相交．那么二者如何相加？如果例如$c_1$与$d_3$是同一个奇异$n$维立方体而其

余的都不相同，则

$$f + g = \sum_{i=2}^{k} a_i c_i + (a_1 + b_3) c_1 + \sum_{j \neq 3} b_j d_j.$$

总之，也是一个“形式”加法．

由奇异立方体到链以至到“循环”和“边缘”，我们处理的都是形式加法．尽管如此，我们仍然说这是一种代数的方法．在现代数学中，这种“代数的方法 ”是极为重要的．

4-23 如果固定一个 R 则 $c_{R,n}(t)$ 是半径为 R 的圆周，但是作逆时针方向转了 n 圈．这里是讲的 $n>0$. 如果 $n<0$，则是依顺时针方向转了 $|n|$ 圈（或者说转了 $-n$ 圈)．现在再让 R 变化，例如从 a 到 b，则得到一个由矩形 $[a,b]\times[0,1]$ 到圆环映射，把圆环覆盖了 n 次．每一个窄条 $[a,b]\times[\frac{k}{n},\frac{k+1}{n}]$ 覆盖一次，但是这个矩形还不是原题中的 $[0,1]^2$. 为此还要再在 $\mathbf{R}$ 轴方向上作一个变换

$$R = a + \rho(b-a), \qquad 0 \leqslant \rho \leqslant 1$$

与此映射复合起来才得到原题的要求：$[0,1]^2 \to \mathbf{R}^2 - 0$. 特别需要注意的是 ∂c 的求法．在图 A-14 左下面我们把一个窄条上的边缘的定向用箭头标出．从 A 到 B（虚线）相当于圆环上由 Q 到 P（下岸)，然后到第二个窄条中又由 B 到 A（虚线）相当于环上由 P 到 Q（上岸)，二者抵消．如此以往只余下最右的由 $\rho=0$ 到 $\rho=1$ 以及最左的由 $\rho=1$ 到 $\rho=0$，又相当于环上由 Q 到 P 和由 P 到 Q，上下岸各是一次，二者又抵消．“抵消”这个说法在几何上含义非常明显，但在代数上如何表示？这就是 $\overrightarrow{PQ}+\overrightarrow{QP}=0$. 但是我们遇到的并非向量 $\overrightarrow{PQ}$ 而是一条链．因此就有了链的“代数运算” $\overrightarrow{QP}=(-1)\overrightarrow{PQ}$，$\overrightarrow{PQ}+(-1)\overrightarrow{PQ}=0$，这正是上题中讲的形式运算．但是现在的 $\overrightarrow{PQ}$ 确是向量，上面的运算又与向量的运算非常相近．那么，链的运算是否就是线性空间的运算？实际上它的内容还要丰富得多．不妨说，这两

个题目正是“代数拓扑学”的一个切入点．下一题的性质也是这样．

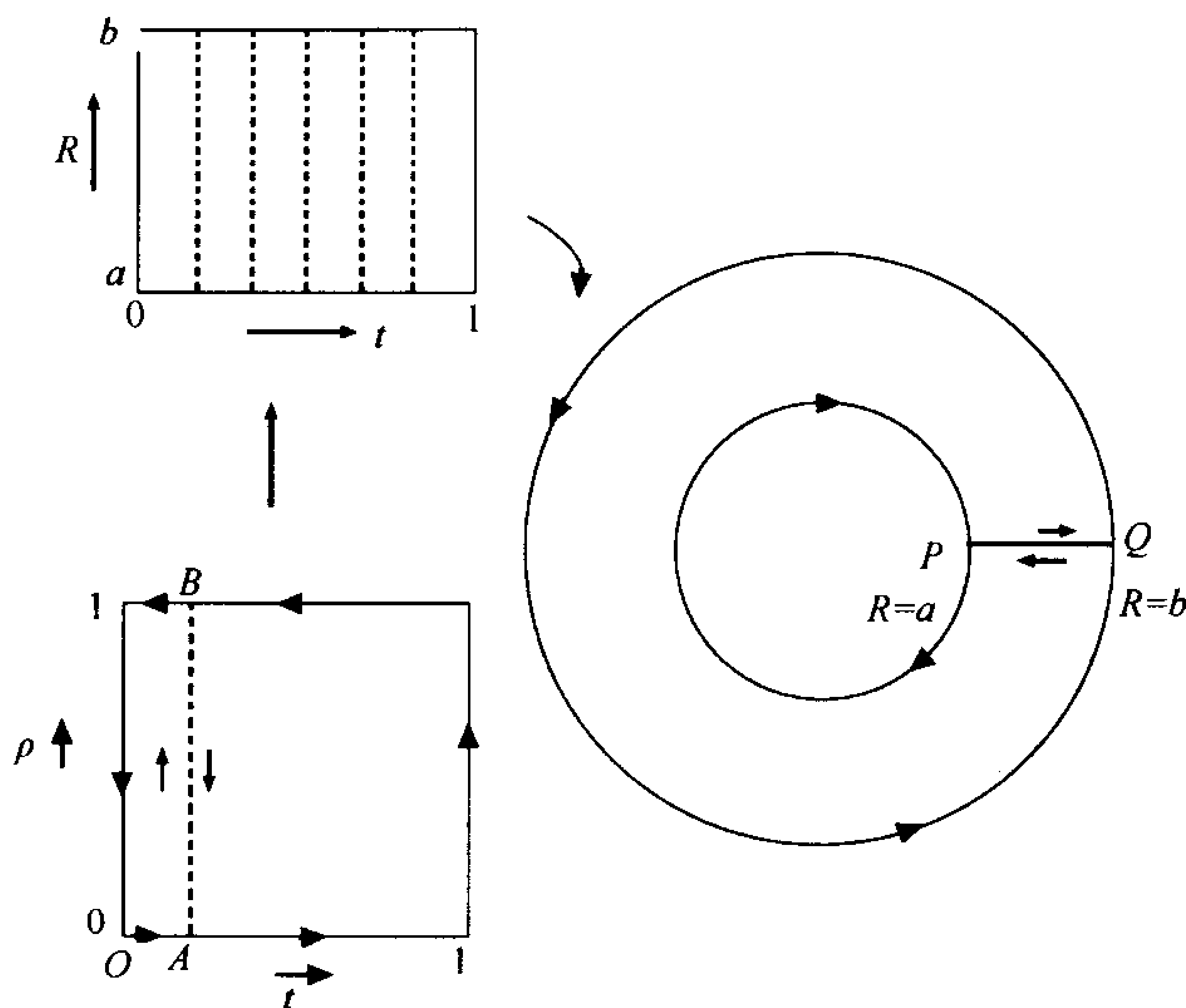


图 A-14

4-24 “c 是一个奇异 1 维立方体，而且 $c(0) = c(1)$” 这句话就是说 c 是一条封闭曲线，0 和 1 是起点与终点（这是同一点 A）的参数值．图 A-15 是一个示意图，

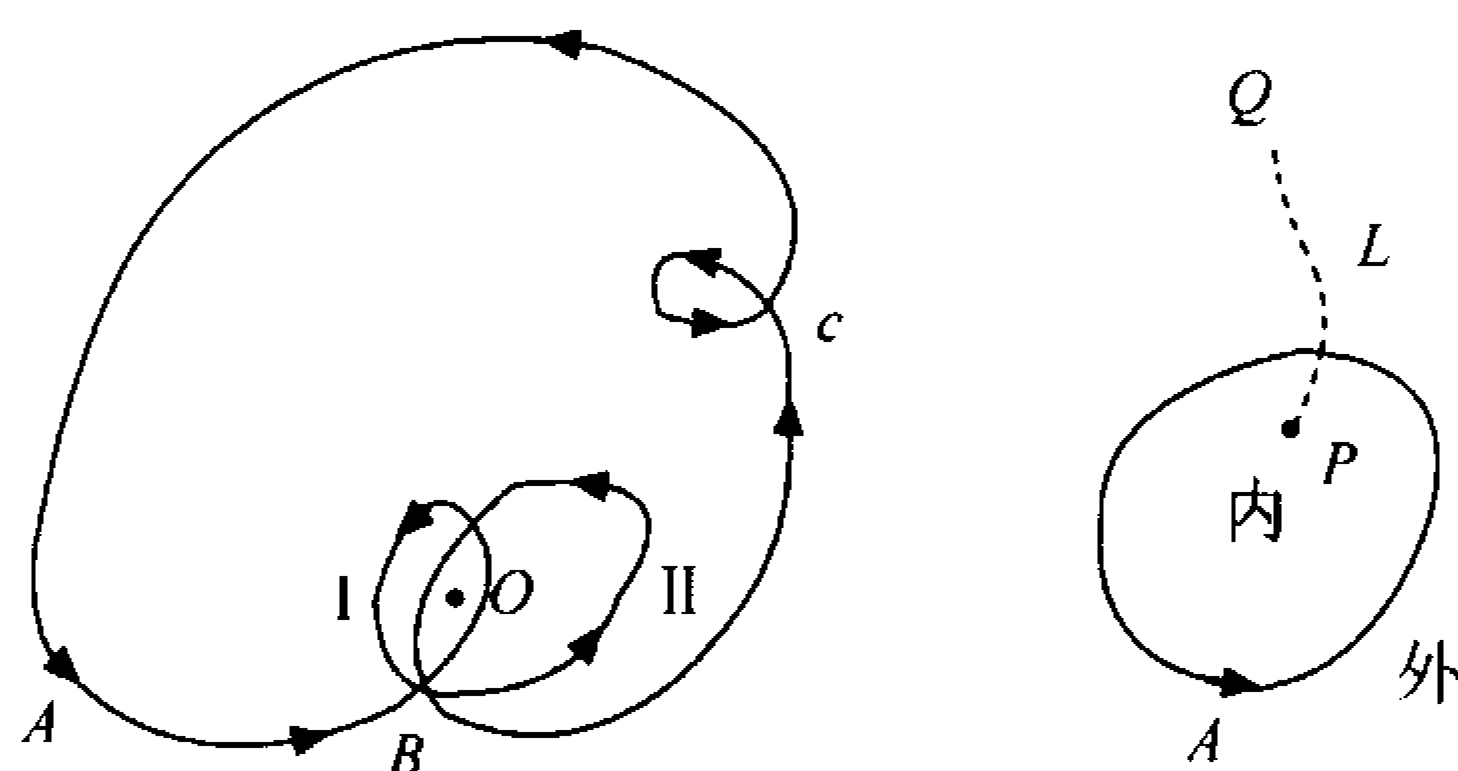


图 A-15

当然，实际情况可能复杂得多．在最简单的情况，这条曲线不自交．不自交的封闭曲线在数学上称为约当曲线（与约当测度的命名来自同一个人）．有一个著名的约当定理：平面上的一条约当曲线必将平面分为内外两部分，若在其内外各取一点 P，Q 并用一条连续曲线 L 连结它们，则 L 必与此曲线相交．这

个定理时常使读者大为困惑：这样十分明显的事还要证明吗？(事实上证明极难)，问题远非如人们想像的那么简单．如图A-15 那样的区域何为其内何为其外？这条曲线 c 在 B 点自交多少次？它绕过 O 点多少圈？而内、外问题我们在学习线积分时一定会遇到它，且在数学的其他分支以及在物理学中它都是很重要的．本书中讲的线积分与我们在通常的微积分课程中所学的，其实大有不同．例如我们就要考虑图 A-15 左边那样的封闭曲线上的线积分，或更复杂的问题还要问例如格林定理这时是否也成立？这是本书的核心．我们过去学的线积分教本，没有一本说不行，但是真正要动手却又束手无策！其中一个关键问题就是如何处理这样复杂的积分路径．本题提出的就是一种比约当定理更好的处理方式．

把 $[0,1]$ 分成许多段 $[t_0,t_1]\cup[t_1,t_2]\cup\cdots\cup[t_n,t_{n+1}]$，并使每个 t_k 均为自交点，并在 (t_k,t_{k+1}) 中没有其他自交点．如图A-16. 用上题的作法即把这个小窄条先变为一个圆 $c_{1,1}$（后面一个指标表示第一个窄条）．把它除去，再作下一个类似的自交点，于是有了两圆（或同一个圆走两圈）$c_{1,2}\cdots$ 仿此到了最后一个自交点 t_{n+1}，它恰好与 t_0 一样对应于起点 $A(t_0,t_{n+1}$ 也是自交点)．把该除去的都除去，最后留下的是图 A-15 左图最外围的 $ABCA$. 这是一条约当曲线，而且是一个平面区域（二维链）c^2 的边缘．所以从 c 中除掉 n 个圆（即同一个圆走 n 次）$c_{1,n}$，用形式运算表为 $c-c_{1,n}=\partial c^2$.

这样的讲法当然有失严格，但是却表明了原题的含意．如果想了解更“严格”的讲法，可以参看本书参考文献［1］第4章第4节（中文译本 137—147 页）．一本更清楚的书是 S. Lang, *Complex Analysis*, 3e, Springer-Verlag, 1994, 3-4 章)．

4-25 本题求证应改为

$$\int_c \omega=\int_{c\circ p} p^*\omega,$$

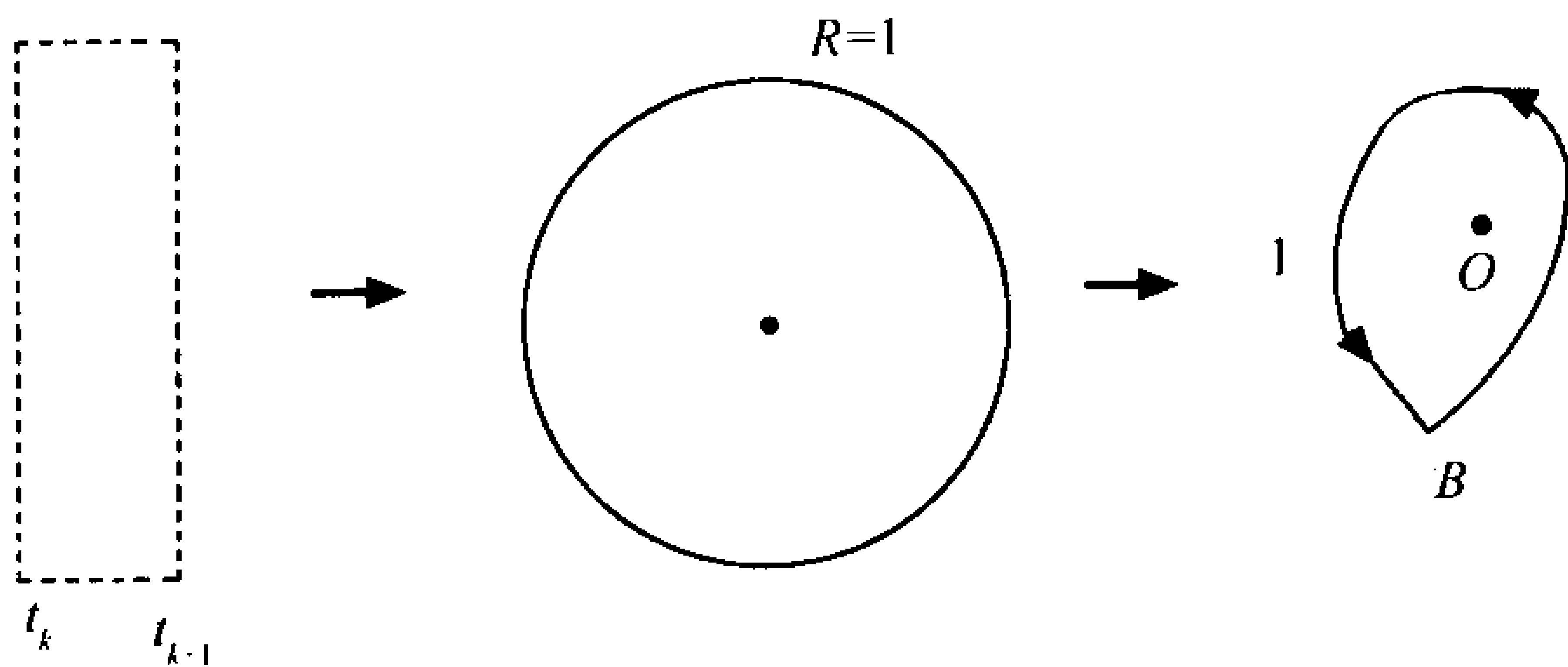


图 A-16

$$\int_{p_*([0,1]^k)} \omega = \int_{[0,1]^k} p^* \omega.$$

它其实就是积分中的变量替换．如果 $\det p' < 0$ 结果会如何？

4-26 本题是习题 4-24 的继续．$\int_{C_{R,n}} \mathrm{d}\theta = 2n\pi$ 易证．如果 $c_{R,n} = \partial c$，而且原点 O 不在二维链 c 中，当有

$$\int_{c_{R,n}} \mathrm{d}\theta = \int_{\partial c} \mathrm{d}\theta = \int_{c} \mathrm{d}^2\theta = 0,$$

这里 n 是整数．

4-27 习题 4-21 告诉我们，在 $\mathbf{R}^2 - 0$ 上，

$$\mathrm{d}\theta = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

是恰当形式，因此 $\mathrm{d}^2\theta = 0$. 由习题 4-24，有

$$\int_{c-c_{R,n}} \mathrm{d}\theta = \int_{c} \mathrm{d}\theta - \int_{c_{R,n}} \mathrm{d}\theta = \int_{\partial c^2} \mathrm{d}\theta = \int_{c^2} \mathrm{d}^2\theta = 0.$$

因此，再由习题 4-26 有

$$\int_{c} \mathrm{d}\theta = 2n\pi .$$

此式左方的积分与 $c_{R,n}$ 之选择无关，而只决定于 c. 所以右方的 $n = \frac{1}{2\pi}\int_{c} \mathrm{d}\theta$ 也只与 c 有关，原题得证．

这个整数 n（即环绕数）刻画了闭曲线 $c:[0,1]\to\mathbf{R}^2-O$ 绕过 O 的次数，因为 $\int_M^P d\theta=\theta$ 表示动径 $\overrightarrow{OP}$（从 M 算起）当 P 绕 c 一周后一共转了多少圈（顺时针方向转一圈算 -1 圈）. 如果 c 十分复杂. 则从图形上直观地想出圈数是很不容易的，所以称为环绕数. 它是一个非常有用的概念，而且可以看作是对约当定理给了一个很方便的表述. 因为如果令 O 连续地微小变动，只要不遇到 c，环绕数也只会连续地变化. 但环绕数是整数，则‘连续变化’就是不变. 因此只要 O 在变动中不遇到 c，环绕数一定是常数. 这样一来，环绕数作为 O 的函数是局部常值函数. 对于约当曲线，很容易看到，当 O 在曲线 c 所围的区域之内，$\overrightarrow{OP}$ 恰好只转一圈，而环绕数为 1. 当 O 在曲线 c 所围的区域之外时，则环绕数为 0. 因此，约当定理就可以陈述为：平面上任一约当曲线 L 必将平面分成两部分，对于一部分中之点，环绕数为 1，这一部分称为内部；对于另一部分中的点环绕数为 0，这一部分称为外部. 如果一点从内部走到外部，它一定再经过曲线 L.

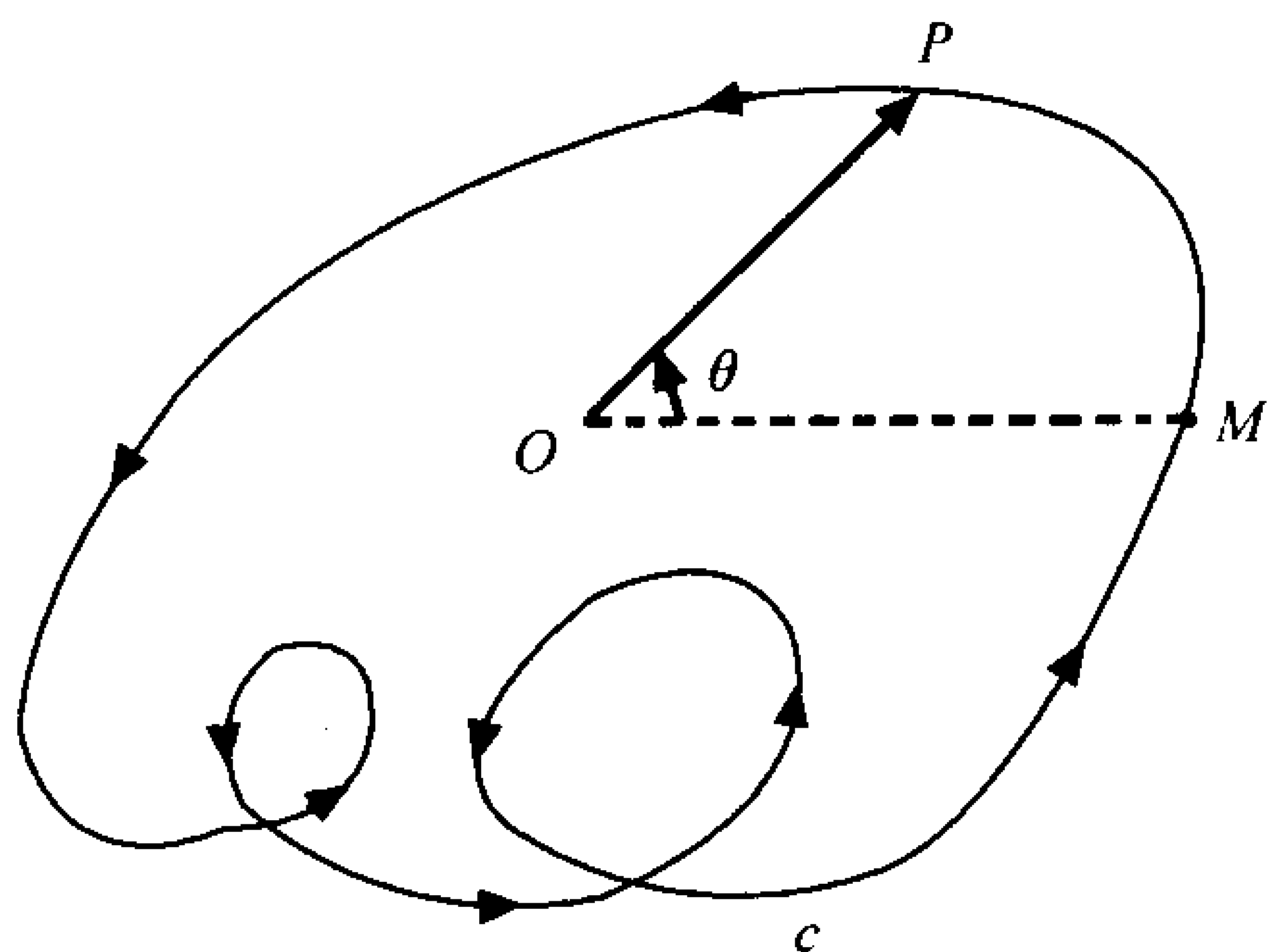


图 A-17

请读者思考一下，图 A-17 中的 c 把平面划分成了几个部分？把 O 移到某一部分中来看各部分之绕数是多少？再思考一下，哪些算内部，哪些算外部？

真正提醒读者注意的是这里的思想与技巧在数学中有重要的应用，下题是其一例.

4-28 代数学的基本定理（以下简记为 FTA）是每个读者都知道的. 它的原始的证明是高斯给出的，可以在一本“很老的”代数教

本中找到，但是它的证明太困难了，而且高斯实际上（不自觉）用了一个基本的拓扑定理——连续函数的中间值定理，以致人们认为高斯的证明“有毛病”．下面的证明突出的一点，就是应用了环绕数，从而真正表明了这个定理的本质．

如原题所示，我们进入复域，引入复的自变量 z，并令 $f(z) = z^n + a_1z^{n-1} + \cdots + a_n$，这里 $a_1,\cdots,a_n$ 均为复数，$n > 0$. 于是证明的基本思想如下：注意 $\mathbf{R}^2$ 就是 $\mathbf{C}$ 平面．$\mathbf{R}^2$ 平面上的曲线（如果用参数表示）$x = x(t), y = y(t)$ 也就是 $x + iy = x(t) + iy(t)$. 令 $z = x + iy$，立刻看到 $\mathbf{R}^2$ 上的实曲线与 $\mathbf{C}$ 平面上的复曲线是一回事．为简单计，不妨用极坐标 $z = re^{i\theta}$，并考虑 z 在半径为 R 的圆周 c_R 上运动，而 $\theta \in [0,2\pi]$ 是一个参数．原书上如果把参数 θ 写成 $\theta = 2\pi t$，则 t 成为参数而在 $[0,1]$ 上变化，这就是原书说的奇异 1 维立方体 $c_{R,t}$. 对 f 我们也有一个新看法，认为它是由 z 所在的 $\mathbf{C}$ 平面到 $w = f(z)$ 所在的 $\mathbf{C}$ 平面．如果 $f(z)$ 有一个根 $z_o = R_0e^{i\theta_0}$，过它作一个圆周如图 A-18 之左，则 f 仍把它映为 w 平面上的封闭曲线而且经过 $0 = f(z_0)$. 原来的圆周 $c_{R,t}$ 现在成了 w 平面上的另一个奇异 1 维立方体，即原书上说的 $c_{R,f} = f\circ c_{R,t}$.

下面回到环绕数的概念．习题4-27 中提到 c 对某一点 $w = 0$ 的环绕数，它来自习题4-24，其中规定 $c \subset \mathbf{R}^2 - 0$. 在我们的情况下，即 w 平面上的曲线 $c_{R,f} = f\circ c_{R,t}$ 不能经过 $w = 0$. 而右图的 $c_{R_0,f}$ 则恰好经过了 $w = 0$，因此在这一条曲线的附近，环绕数的变化会出现怪异．这就是本题的要害所在．那么 $c_{R,f}$ 对 0 的环绕数怎样变化呢?

(a) 指出当 R 充分大时，因为 $\partial c = c_{R,f} - c_{R,n}$，从而由习题4-27，它的环绕数是 $\frac{1}{2\pi}\int_{cR,f} \mathrm{d}\theta = n\, c_{R,f}$（$f(z)$ 对 z 之次数）. 原题找出了习题4-24 所说的 2 维链 c^2，但通常可用一个更接近微积分的证法．令 $|z| \geqslant R$（注意，这里讲的是 z 平面上的事），则只要 R 充分大，必有：

$$|f(z)|\geqslant|z|^n-(|a_1|\ |z|^{n-1}+\cdots+|a_n|)$$
$$=|z|^n[1-(\frac{|a_1|}{|z|}+\cdots+\frac{|a_n|}{|z|^{n-1}})]$$
$$=R^n[1-\sum_{h=1}^{n}\frac{|a_h|}{R^h}]\geqslant\frac{1}{2}R^n.$$

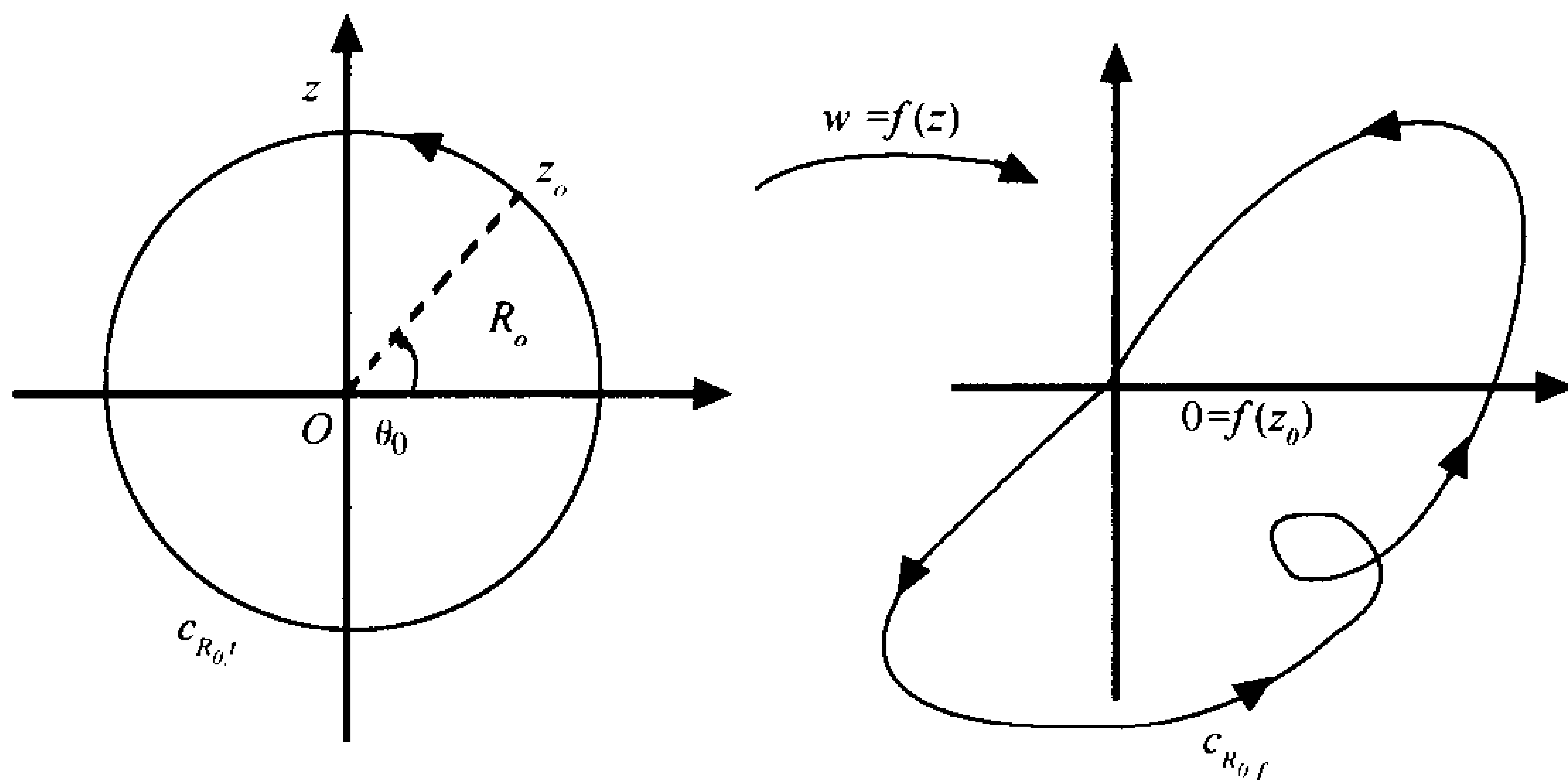


图 A-18

就是说 w 平面上的曲线 $c_{R,f}$ 必在半径为 $\frac{1}{2}R^n$ 的圆外. 再作一个更大的圆 $|z|=R_1$, 则 $c_{R,f}$ 又在 w 平面上的圆 $|w|=R_1{}^n$ 之内（见图 A-19）. 这个阴影区域就是上面说的 c^2.

由此可知当 R 充分大时 $c_{R,f}$ 关于 $w=0$ 之环绕数为 n.

(b) 下面要看另一个极端如图 A-20. 为此，我们不妨设 $a_n=0$, 因为 $a_n=0$ 时, $f(z)=0$ 自然有一个根 $z_o=0$, 现在让 R 非常小，则 $c_{R,t}$ 是 z 平面上以 $z=0$ 为心的“小”圆. 而其在 w 平面上的像 $c_{R,f}$ 则是位于 $a_n(\neq 0)$ 附近的一条闭曲线，不论它是否绕过 a_n, 不论其构造如何复杂，它总不能绕过 $w=0$. 所以 $c_{R,f}$ 这时关于 $w=0$ 的环绕数成为 0.

那么怎样把这两个极端情况（R 极大，或 R 极小）联系起来呢，引入一个参数 $\rho=\lambda R$, R 是很大的数, $\lambda\in[0,1]$, 则

$\lambda=0$ 和 1 对应于很大的 $c_{R,t}$ 与很小的 $c_{R,t}$ 两个极端，再加上每个圆 $c_{R,t}$ 上的参数 $\theta=2\pi t$. 不过 $t=0$ 和 $t=1$ 即 $\theta=0,2\pi$ 是同一个点. 如果令 $F_{\lambda,t}(z)=f(\lambda Re^{i\theta})$，则 $R\approx0$，$\lambda=1$ 对应于很小的 $c_{R,f}$ 与很大的 $c_{R,f}$，这样就把两个极端情况联系起来了. 对固定的 λ，即得

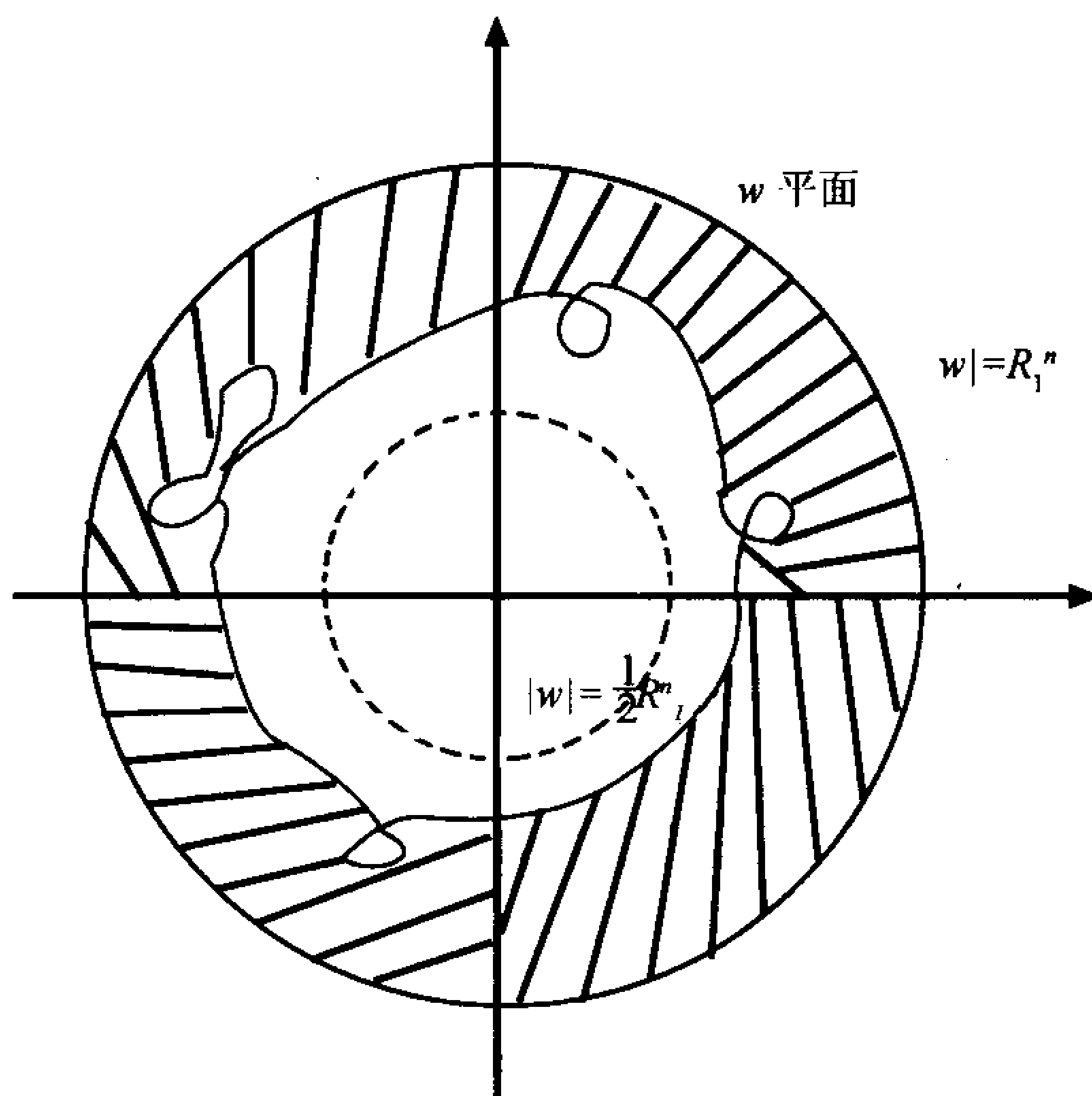


图 A-19

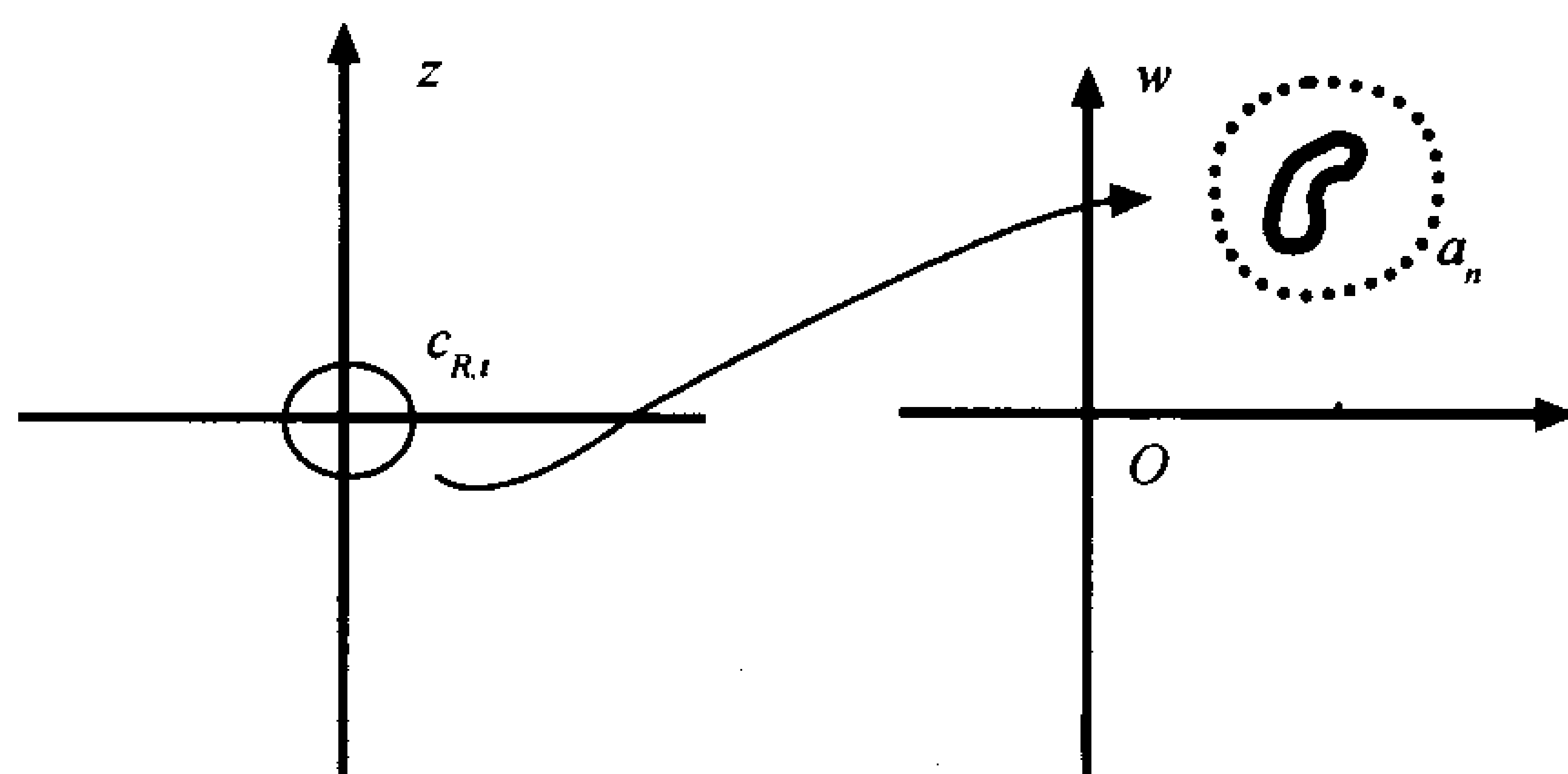


图 A-20

$$f(\lambda z) = (\lambda z)^n + a_1(\lambda z)^{n-1} + \cdots + a_n.$$

它映 z 平面上的圆 $c_{R,t}$ 为 w 平面上一条曲线.

现在我们可以用反证法来证明 FTA 了. 如果 $f(z)$ 在 $\mathbf{C}$ 上没有零点, 则不可能有任一个 $z = \lambda Re^{i\theta}$ 使 $f(z) = 0$. 这样曲线由很大的 $c_{R,f}$ 变成很小的 $c_{R,f}$ 时, 环绕数就一定连续变化 (见习题 4-27 之解答说明), 那么就不可能由 R 很大时的环绕数 n 变为 $R \to 0$ 时的环绕数 0. 这个矛盾证明了 FTA.

以上的证明是不严格的. 为了要做到严格, 就需要代数拓扑学中的同伦理论.

第 4 章 4-28 后面各个题一部分是关于同伦理论的, 另一部分是对复变函数论的介绍, 也是利用了代数拓扑学的基本思想. 原书的意图是希望读者能由此进入其他数学分支, 所以我们就不再讲解了.

5. 流形上的积分

5-3 (a) 不妨只考虑 $x_0 \in A$ 的边界的邻域, 而且因为 A 的边界是 $(n-1)$ 维流形, 不妨取一个局部坐标系使 x_0 即为原点, 而 A 的边界为 $x_n = 0$. 于是 $x_n > 0$ (或 $x_n < 0$) 中必有 A 之点, 且为内点. 于是它又有一个开邻域 (开长方体) 全在 A 中. 取这些开长方体之并即知 $x_n = 0$ 附近的上半空间 $x_n > 0$ 应全在 A 内. 对 $x_n < 0$ 亦作类似处理. 这样就可以区别两个情况:

(1) $x_n > 0$ (或 $x_n < 0$) 中 $x_n = 0$ 附近全在 A 中, 而 $x_n < 0$(或 $x_n > 0$) 中 $x_n = 0$ 附近全不在 A 中. 这时, A 局部地可以与上半空间 $x_n > 0$ 同胚, 而 A 成为有边流形, 局部地 ∂A 即为 $x_n = 0$.

(2) $x_n > 0$ 与 $x_n < 0$ 中 $x_n = 0$ 的附近全在 A 中, 但 $x_n = 0$ 在 A 的边界中. 这时局部地 A 的边界即为 $x_n = 0$, 但它不是 A 的边缘. 总之 $N = A \cup$ (A 的边界). 或者尽含 0 的一个完全的领域,

或者只含半空间 $x_n \geqslant 0$ 的一部分．则 A 的所有边界点均作此处理后，即知或者 N 没有边缘：$\partial N = \emptyset$，或者 N 是有边流形．但是我们可以把一般的流形也看作有边流形的特例（见习题 5-19），这样原题得证．如果不把一般的流形看作有边流形的特例，则本题改成加一个条件：$N = A \cup (A$ 之边界$) \neq \mathbf{R}^n$ 就对了，对此下面有一反例：

原书举了一个例：$A = \{x \in \mathbf{R}^n : |x| < 1$ 或 $1 < |x| < 2\}$，即一个开球与一个开环之并．A 的边界由两部分组成：球面 $\{x: |x| = 1\}$ 与外环面 $\{x: |x| = 2\}$．后者是情况(1)，而前者是情况(2)．$\partial N = \{x: |x| = 2\}$．但若把 A 改为 $A = \{x \in \mathbf{R}^n : |x| < 1$ 或 $|x| > 1\}$ 即知有时 N 确实是无边的．这时 $N = A \cup (A$ 之边界$) = \mathbf{R}^n$.

5-6 f 的图像是 $\mathbf{R}^{n+m} = \mathbf{R}^n_x \times \mathbf{R}^m{}_y$ 的一个子集合，而 f 则是一个映射．把一个映射与图像区别开来是现代数学中常用的思想．

本题其实是定理 5-2 的一个推论．条件(c)中的 U 现在取为 $f(x)$ 的定义域，它也就是 W 的一个开子集．(c)中的 $\mathbf{R}^k$ 现在成了 $\mathbf{R}^n$，而可微映射 f 之像 $\mathbf{R}^n$ 现在改成 $\mathbf{R}^{n+m}$．(c)中的可微映射 f 并不是本题中的 f，而是 $y = F(x)$ 如图 A-21.

$$F\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}.$$

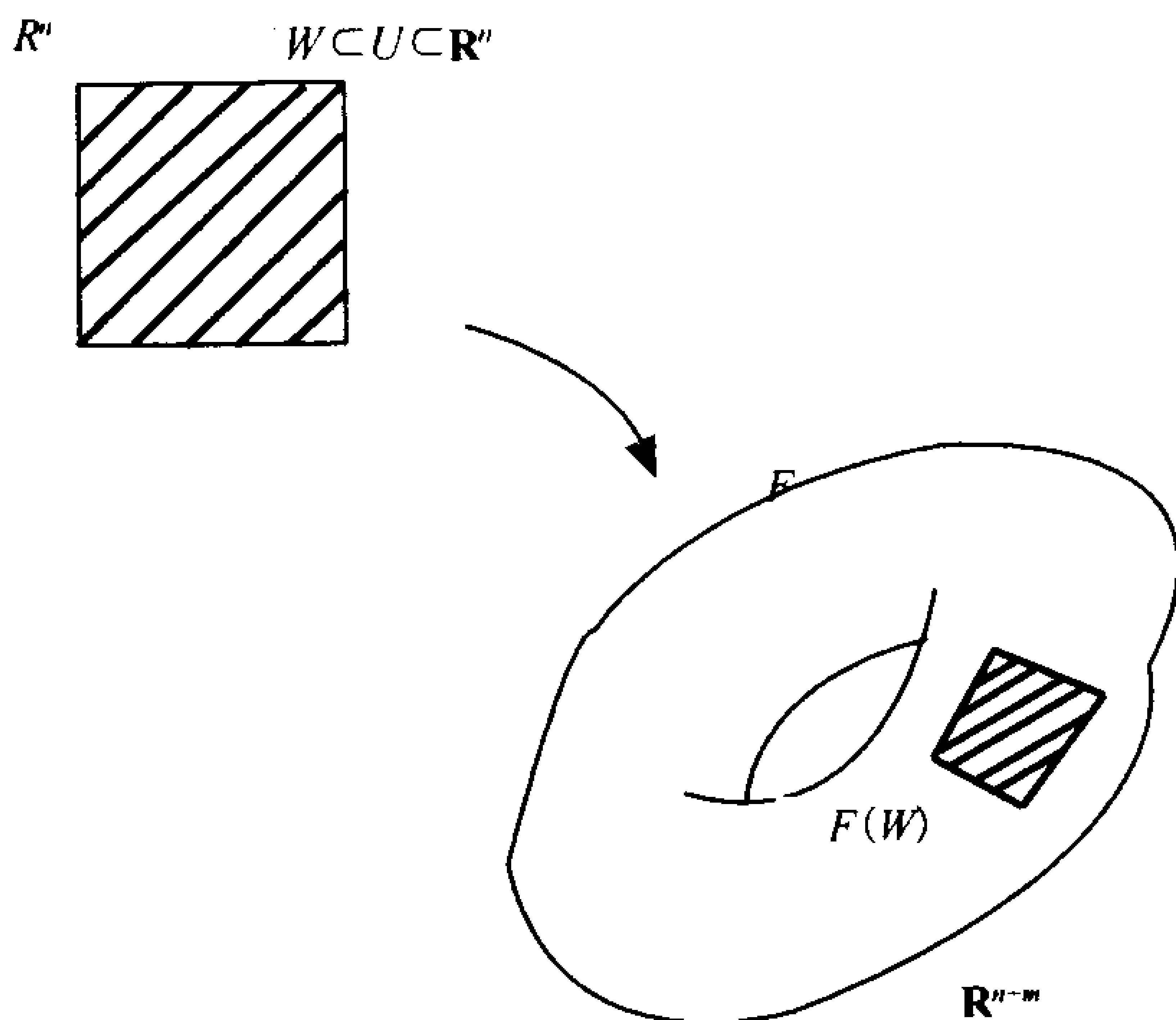


图 A-21

或写成

$$z_i = x_i, \qquad 1 \leqslant i \leqslant n$$

$$z_j = f_{j-n}(x_1, \cdots, x_n), \quad n+1 \leqslant j \leqslant n+m$$

这样的 F 在 $F(w)$ 与 W 之间建立了一个连续的 1-1 对应．因为对 $F(w)$ 中的任一点 y，其前 n 个坐标就是 w 中的 x 点．这个 F 自然是可微的．而且其导数是一个 $(n+m) \times n$ 阶矩阵：

$$F'(x) = \begin{pmatrix} I \\ \dfrac{\partial f}{\partial x} \end{pmatrix},$$

I 是 n 阶单位矩阵，故 $F'(x)$ 之秩为 n（即 5-2 定理中的 k）．

上面讲的 $F(w) \to w$ 的 1-1 连续性可以在取 W 为 U 之充分小开子集后用隐函数定理得出．所以这个结果其实是局部性的，但这并不妨碍结论，因为定理 5-2 本身就是局部性的．

微分流形的定义是很清楚的．但若用它来判断 $\mathbf{R}^n$ 的子集 M 是否流形，并不方便．方便的一个作法是应用定理 5-1，简单地说：流形就是“方程组” $g(x) = 0$ 的零点之集，或者说，流形

就是 $g(x)=0$ 的轨迹．因此我们常见的曲线，曲面都是流形．另一个作法是应用定理 5-2：流形就是由 $\mathbf{R}^k$ 之某个开集到一个更高维空间 $\mathbf{R}^n$ 之像．这其实就是把由方程 $g(x)=0$ 定义的流形说成是用它的参数方程 $F: x \mapsto z$ 来表示．所以这两个定理既直观又重要．在初学时容易误会之处一是各个空间的维数．例如定理 5-2 讲的 $\mathbf{R}^n$ 之维数并不是“自变量”的维数，F 之像的维数也不是函数 $y=f(x)$ 之函数值空间的维数，而是 $y=f(x)$ 之自变量与函数值两个维数之和：$n+m$．另一个容易忽略之处在于这两个定理都有关于秩的条件．

5-8 （b）如果 M 不是闭的，则其边界点可能不在 M 中，因此可能不是 ∂M．例如令 M 为 $\mathbf{R}^n$ 中的开单位球 $\{x: |x| < 1\}$．则（M 之边界）$=\{x: |x| < 1\}$ 是单位球面，它在 M 中当然就不会是 ∂M．其实这里的 M 不是有边流形，这是很容易误会的．

5-9 这个题目的直观含意是很清楚的．如果要作一个曲面过其上一点的切平面，可以先作过此点的“许多”曲线，并作它们过此点的切线．这些切线在此点的切向量除了在一些特异的点以外，必定构成一个平面，即所求的切平面．要除去的一些特异点例如圆锥的顶点，都是所谓“奇点”．图 5-5 可以给出比较直观的理解．现在把这个作法引申到一般的 k 维流形 M．注意到流形上是没有奇点的，所以不会遇到上面讲的圆锥顶点那样的情况．

以下我们将采用定理 5-2 中的记号（注意，条件（c）中是(2)，$f'(y)$ 之秩为 k 即流形 M 之维数，就是上面说的流形上没有奇点）以及图 5-2．示意图如图 A-22．我们现在要讨论 M 在 x 点处的切空间，所以 $x \in M \cap U$ 是固定的，以下记为 a．在 a 附近选取局部坐标 $z=(z_1,\cdots,z_n)$，而 M 被映为 $W \subset \mathbf{R}^k$，即 M 上之点适合 $z_{k+1}=\cdots=z_n=0$．现在过 a 点作 $M \cap U$ 的曲线 $z_i=z_i(t)$，$t \in (-1,1)$，$i=1,2,\cdots,n$．不妨设 $t=0$ 对应于 a 点：$a_i=z_i(0)$．如果有两条这样的曲线 $z_i(t)$ 和 $\tilde{z}_i(t)$，则它们在 a 点相切

的充分必要条件是$z_i'(0)=\tilde{z}_i'(0)=\lambda_i$. 现在按相切关系把这些曲线分成等价类. 则每一个等价类对应于同一个向量$\Lambda=(\lambda_1,\cdots,\lambda_n)$, 称为过$a$点的一个切向量. 读者可能会问, 如果同一条曲线采用了另一个参数τ: $t=t(\tau)$, 而且$\frac{\mathrm{d}t}{\mathrm{d}\tau}\neq 0$(请读者考虑$\neq 0$这一条件意味着什么), 则曲线变成$z_i=z_i(t(\tau))=\mathbf{Z}_i(\tau)$, 而相应的切向量成为$\left.\frac{\mathrm{d}z_i}{\mathrm{d}\tau}\right|_a=z_i'(\tau)|_a=z_i'(0)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)_a=\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)_a\lambda_i$. 所以现在的切向量不是$\Lambda$而是$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)_a\Lambda$. 它与$\Lambda$平行(注意$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)_a\neq 0$), 但并不是同一个切向量. 我们时常以为切向量是斜率概念的推广, 这并不准确. 因为平行的向量虽然不同, 却都有相同斜率. 准确些说, 切向量是速度向量的推广. 如果时间尺度变了(即重新取时间参数τ), 速度方向虽未变, 大小却变了. 这个问题在我们引入向量空间概念后就没有困难了, 因为两个向量Λ与$c\Lambda$虽然不同, 但仍是同一向量空间之元. 读者们又会要问, 向量的加法怎样反映在曲线的等价类的“加法”上? 回答这个问题之前, 先问另一个问题. 以上我们作的是过a点的U中的曲线, 而我们需要的是在$M\cap U$上的曲线. 怎样从一般的曲线中划分出位于M上的那一些? 它们相应的切向量是否成为一个k维子空间? 这也就是说, 这些切向量是否构成一个“平面”?

为此我们注意到, 当f之秩为k时, 由f及h诱导出来的f_*, h_*均为1-1的线性变换. 所以M上的相切曲线所成的等价类——用一个向量Λ表示将被映为$\xi=h_*\Lambda$(记号上有不清楚处请读者自行改进). 反之$\mathbf{R}_a^k$中的向量ξ亦必被映为$\Lambda=f_*\xi$, 而且因为f_*是线性同构(即1-1线性变换), 所以Λ之集合也是一个k维线性空间即切空间M_a, 所以$M_a=\{\Lambda\}$.

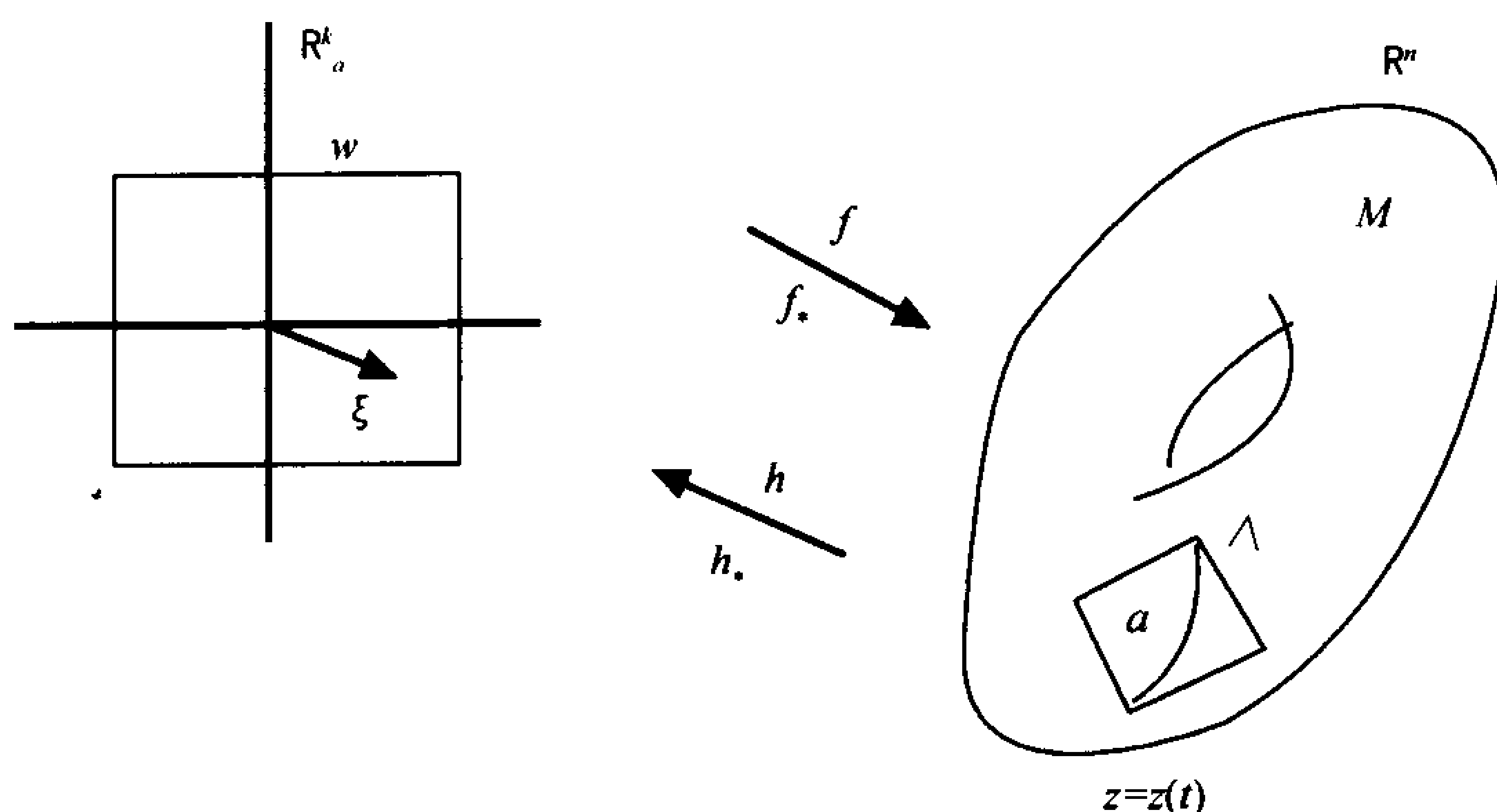


图 A-22

5-12 这个结果称为拓展定理，即一个微分流形 M 上的可微向量场 F，必可拓展到包含 M 的某个开集 A 上去．这个似乎自明的结论证明起来相当复杂，需要查找有关的专门著作．例如有 seeley 的拓展定理，它指出，若 M 的边缘相当规则，则每一个定义在 M 上的 C^∞ 函数 $f(x)$，必可找到一个定义在 $\mathbf{R}^n$ 上的 C^∞ 函数 $F(x)$，使当 $x\in M$ 时，$F(x)=f(x)$．关于这个定理的准确提法和证明，可以参看 J. Chazarain and A. Piriou, *Introduction to the Theory of Partial Differential Equations*. 1982.

5-13 (a)原题记号与定理 5-1 略有区别．这里的 U 应该就是定理 5-1 中的 A. 这个题目的几何意义如下．正如 5-6 中的 f 是一个低维流形 $\mathbf{R}^n$ 在更高维的 $\mathbf{R}^{n+m}$ 中的“嵌入”，现在则把一个流形表为 $g^{-1}(0)$ 而变成由较高维的 $\mathbf{R}^n$ 到较低维的 $\mathbf{R}^{n-p}$ 上的“投影”．如图 A-23.

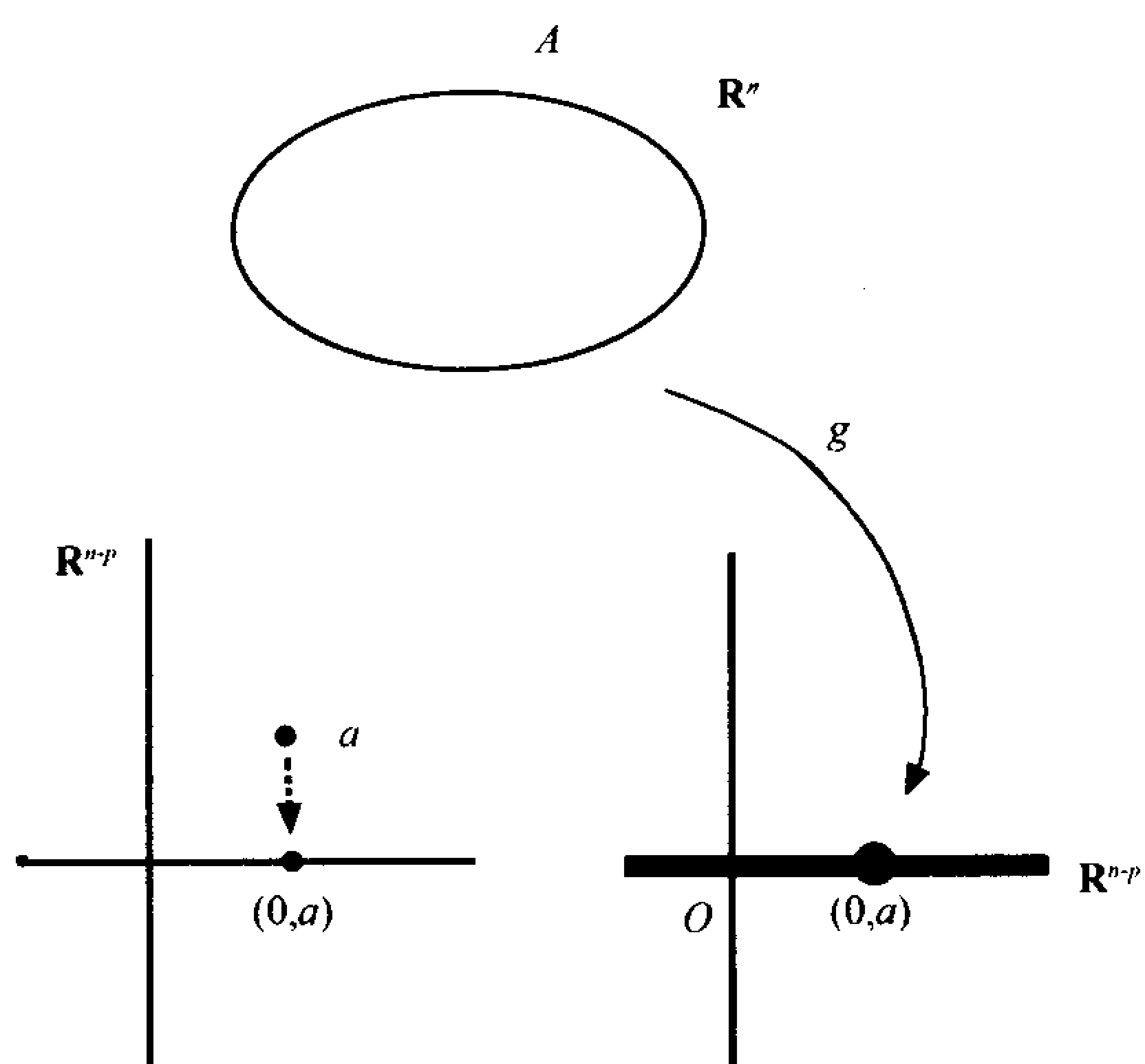


图 A-23

1

Functions on Euclidean Space

NORM AND INNER PRODUCT

Euclidean *n*-space $\mathbf{R}^n$ is defined as the set of all n-tuples $(x^1, \ldots, x^n)$ of real numbers x^i (a "1-tuple of numbers" is just a number and $\mathbf{R}^1 = \mathbf{R}$, the set of all real numbers). An element of $\mathbf{R}^n$ is often called a point in $\mathbf{R}^n$, and $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$ are often called the line, the plane, and space, respectively. If x denotes an element of $\mathbf{R}^n$, then x is an n-tuple of numbers, the ith one of which is denoted x^i; thus we can write

$$x = (x^1, \ldots, x^n).$$

A point in $\mathbf{R}^n$ is frequently also called a vector in $\mathbf{R}^n$, because $\mathbf{R}^n$, with $x + y = (x^1 + y^1, \ldots, x^n + y^n)$ and $ax = (ax^1, \ldots, ax^n)$, as operations, *is* a vector space (over the real numbers, of dimension n). In this vector space there is the notion of the length of a vector x, usually called the **norm** $|x|$ of x and defined by $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$. If $n = 1$, then $|x|$ is the usual absolute value of x. The relation between the norm and the vector space structure of $\mathbf{R}^n$ is very important.

1-1 Theorem. *If $x,y \in \mathbf{R}^n$ and $a \in \mathbf{R}$, then*
(1) $|x| \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$.
(2) $|\sum_{i=1}^n x^i y^i| \leq |x| \cdot |y|$; *equality holds if and only if x and y are linearly dependent.*
(3) $|x + y| \leq |x| + |y|$.
(4) $|ax| = |a| \cdot |x|$.

Proof

(1) is left to the reader.
(2) If x and y are linearly dependent, equality clearly holds. If not, then $\lambda y - x \neq 0$ for all $\lambda \in \mathbf{R}$, so

$$0 < |\lambda y - x|^2 = \sum_{i=1}^n (\lambda y^i - x^i)^2$$
$$= \lambda^2 \sum_{i=1}^n (y^i)^2 - 2\lambda \sum_{i=1}^n x^i y^i + \sum_{i=1}^n (x^i)^2.$$

Therefore the right side is a quadratic equation in λ with no real solution, and its discriminant must be negative. Thus

$$4\left(\sum_{i=1}^n x^i y^i\right)^2 - 4\sum_{i=1}^n (x^i)^2 \cdot \sum_{i=1}^n (y^i)^2 < 0.$$

(3)
$$\begin{aligned} |x + y|^2 &= \textstyle\sum_{i=1}^n (x^i + y^i)^2 \\ &= \textstyle\sum_{i=1}^n (x^i)^2 + \sum_{i=1}^n (y^i)^2 + 2\sum_{i=1}^n x^i y^i \\ &\leq |x|^2 + |y|^2 + 2|x| \cdot |y| \qquad \text{by (2)} \\ &= (|x| + |y|)^2. \end{aligned}$$

(4) $|ax| = \sqrt{\sum_{i=1}^n (ax^i)^2} = \sqrt{a^2 \sum_{i=1}^n (x^i)^2} = |a| \cdot |x|$. ▮

The quantity $\sum_{i=1}^n x^i y^i$ which appears in (2) is called the **inner product** of x and y and denoted $\langle x,y \rangle$. The most important properties of the inner product are the following.

1-2 Theorem. *If x, x_1, x_2 and y, y_1, y_2 are vectors in $\mathbf{R}^n$ and $a \in \mathbf{R}$, then*

(1) $\langle x,y \rangle = \langle y,x \rangle$ *(symmetry).*

(2) $\langle ax,y\rangle = \langle x,ay\rangle = a\langle x,y\rangle$ *(bilinearity).*
$\langle x_1 + x_2, y\rangle = \langle x_1,y\rangle + \langle x_2,y\rangle$
$\langle x, y_1 + y_2\rangle = \langle x,y_1\rangle + \langle x,y_2\rangle$

(3) $\langle x,x\rangle \geq 0$, and $\langle x,x\rangle = 0$ if and only if $x = 0$ *(positive definiteness).*

(4) $|x| = \sqrt{\langle x,x\rangle}$.

(5) $\langle x,y\rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$ *(polarization identity).*

Proof

(1) $\langle x,y\rangle = \Sigma_{i=1}^n x^i y^i = \Sigma_{i=1}^n y^i x^i = \langle y,x\rangle$.

(2) By (1) it suffices to prove

$$\langle ax,y\rangle = a\langle x,y\rangle,$$
$$\langle x_1 + x_2, y\rangle = \langle x_1,y\rangle + \langle x_2,y\rangle.$$

These follow from the equations

$$\langle ax,y\rangle = \sum_{i=1}^n (ax^i)y^i = a\sum_{i=1}^n x^i y^i = a\langle x,y\rangle,$$
$$\langle x_1 + x_2, y\rangle = \sum_{i=1}^n (x_1{}^i + x_2{}^i)y^i = \sum_{i=1}^n x_1{}^i y^i + \sum_{i=1}^n x_2{}^i y^i$$
$$= \langle x_1,y\rangle + \langle x_2,y\rangle.$$

(3) and (4) are left to the reader.

(5) $$\frac{|x+y|^2 - |x-y|^2}{4}$$
$$= \tfrac{1}{4}[\langle x+y, x+y\rangle - \langle x-y, x-y\rangle] \qquad \text{by (4)}$$
$$= \tfrac{1}{4}[\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle - (\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle)]$$
$$= \langle x,y\rangle. \quad \blacksquare$$

We conclude this section with some important remarks about notation. The vector $(0, \ldots ,0)$ will usually be denoted simply 0. The **usual basis** of $\mathbf{R}^n$ is $e_1, \ldots ,e_n$, where $e_i = (0, \ldots ,1, \ldots ,0)$, with the 1 in the ith place. If $T: \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, the matrix of T with respect to the usual bases of $\mathbf{R}^n$ and $\mathbf{R}^m$ is the $m \times n$ matrix $A = (a_{ij})$, where $T(e_i) = \Sigma_{j=1}^m a_{ji}e_j$ —the coefficients of $T(e_i)$

appear in the ith *column* of the matrix. If $S\colon \mathbf{R}^m \to \mathbf{R}^p$ has the $p \times m$ matrix B, then $S \circ T$ has the $p \times n$ matrix BA [here $S \circ T(x) = S(T(x))$; most books on linear algebra denote $S \circ T$ simply ST]. To find $T(x)$ one computes the $m \times 1$ matrix

$$\begin{pmatrix} y^1 \\ \cdot \\ \cdot \\ \cdot \\ y^m \end{pmatrix} = \begin{pmatrix} a_{11}, & \ldots & ,a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{m1}, & \ldots & ,a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{pmatrix};$$

then $T(x) = (y^1, \ldots ,y^m)$. One notational convention greatly simplifies many formulas: if $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, then (x,y) denotes

$$(x^1, \ldots ,x^n,y^1, \ldots ,y^m) \in \mathbf{R}^{n+m}.$$

Problems. **1-1.*** Prove that $|x| \leq \sum_{i=1}^n |x^i|$.

1-2. When does equality hold in Theorem 1-1(3)? *Hint:* Re-examine the proof; the answer is not "when x and y are linearly dependent."

1-3. Prove that $|x - y| \leq |x| + |y|$. When does equality hold?

1-4. Prove that $\big|\, |x| - |y| \,\big| \leq |x - y|$.

1-5. The quantity $|y - x|$ is called the **distance** between x and y. Prove and interpret geometrically the "triangle inequality": $|z - x| \leq |z - y| + |y - x|$.

1-6. Let f and g be integrable on $[a,b]$.

(a) Prove that $\left|\int_a^b f \cdot g\right| \leq (\int_a^b f^2)^{\frac{1}{2}} \cdot (\int_a^b g^2)^{\frac{1}{2}}$. *Hint:* Consider separately the cases $0 = \int_a^b (f - \lambda g)^2$ for some $\lambda \in \mathbf{R}$ and $0 < \int_a^b (f - \lambda g)^2$ for all $\lambda \in \mathbf{R}$.

(b) If equality holds, must $f = \lambda g$ for some $\lambda \in \mathbf{R}$? What if f and g are continuous?

(c) Show that Theorem 1-1(2) is a special case of (a).

1-7. A linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **norm preserving** if $|T(x)| = |x|$, and **inner product preserving** if $\langle Tx,Ty \rangle = \langle x,y \rangle$.

(a) Prove that T is norm preserving if and only if T is inner-product preserving.

(b) Prove that such a linear transformation T is 1-1 and T^{-1} is of the same sort.

1-8. If $x,y \in \mathbf{R}^n$ are non-zero, the **angle** between x and y, denoted $\angle(x,y)$, is defined as arccos $(\langle x,y \rangle / |x| \cdot |y|)$, which makes sense by Theorem 1-1(2). The linear transformation T is **angle preserving** if T is 1-1, and for $x,y \neq 0$ we have $\angle(Tx,Ty) = \angle(x,y)$.

(a) Prove that if T is norm preserving, then T is angle preserving.

(b) If there is a basis $x_1, \ldots, x_n$ of $\mathbf{R}^n$ and numbers $\lambda_1, \ldots, \lambda_n$ such that $Tx_i = \lambda_i x_i$, prove that T is angle preserving if and only if all $|\lambda_i|$ are equal.

(c) What are all angle preserving $T: \mathbf{R}^n \to \mathbf{R}^n$?

1-9. If $0 \leq \theta < \pi$, let $T: \mathbf{R}^2 \to \mathbf{R}^2$ have the matrix $\begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix}$. Show that T is angle preserving and if $x \neq 0$, then $\angle(x,Tx) = \theta$.

1-10.* If $T: \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \leq M|h|$ for $h \in \mathbf{R}^m$. *Hint:* Estimate $|T(h)|$ in terms of $|h|$ and the entries in the matrix of T.

1-11. If $x,y \in \mathbf{R}^n$ and $z,w \in \mathbf{R}^m$, show that $\langle(x,z),(y,w)\rangle = \langle x,y\rangle + \langle z,w\rangle$ and $|(x,z)| = \sqrt{|x|^2 + |z|^2}$. Note that (x,z) and (y,w) denote points in $\mathbf{R}^{n+m}$.

1-12.* Let $(\mathbf{R}^n)^*$ denote the dual space of the vector space $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, define $\varphi_x \in (\mathbf{R}^n)^*$ by $\varphi_x(y) = \langle x,y\rangle$. Define $T: \mathbf{R}^n \to (\mathbf{R}^n)^*$ by $T(x) = \varphi_x$. Show that T is a 1-1 linear transformation and conclude that every $\varphi \in (\mathbf{R}^n)^*$ is φ_x for a unique $x \in \mathbf{R}^n$.

1-13.* If $x,y \in \mathbf{R}^n$, then x and y are called **perpendicular** (or **orthogonal**) if $\langle x,y\rangle = 0$. If x and y are perpendicular, prove that $|x + y|^2 = |x|^2 + |y|^2$.

SUBSETS OF EUCLIDEAN SPACE

The closed interval $[a,b]$ has a natural analogue in $\mathbf{R}^2$. This is the **closed rectangle** $[a,b] \times [c,d]$, defined as the collection of all pairs (x,y) with $x \in [a,b]$ and $y \in [c,d]$. More generally, if $A \subset \mathbf{R}^m$ and $B \subset \mathbf{R}^n$, then $A \times B \subset \mathbf{R}^{m+n}$ is defined as the set of all $(x,y) \in \mathbf{R}^{m+n}$ with $x \in A$ and $y \in B$. In particular, $\mathbf{R}^{m+n} = \mathbf{R}^m \times \mathbf{R}^n$. If $A \subset \mathbf{R}^m$, $B \subset \mathbf{R}^n$, and $C \subset \mathbf{R}^p$, then $(A \times B) \times C = A \times (B \times C)$, and both of these are denoted simply $A \times B \times C$; this convention is extended to the product of any number of sets. The set $[a_1,b_1] \times \cdots \times [a_n,b_n] \subset \mathbf{R}^n$ is called a **closed rectangle** in $\mathbf{R}^n$, while the set $(a_1,b_1) \times \cdots \times (a_n,b_n) \subset \mathbf{R}^n$ is called an **open rectangle.** More generally a set $U \subset \mathbf{R}^n$ is called **open** (Figure 1-1) if for each $x \in U$ there is an open rectangle A such that $x \in A \subset U$.

A subset C of $\mathbf{R}^n$ is **closed** if $\mathbf{R}^n - C$ is open. For example, if C contains only finitely many points, then C is closed.

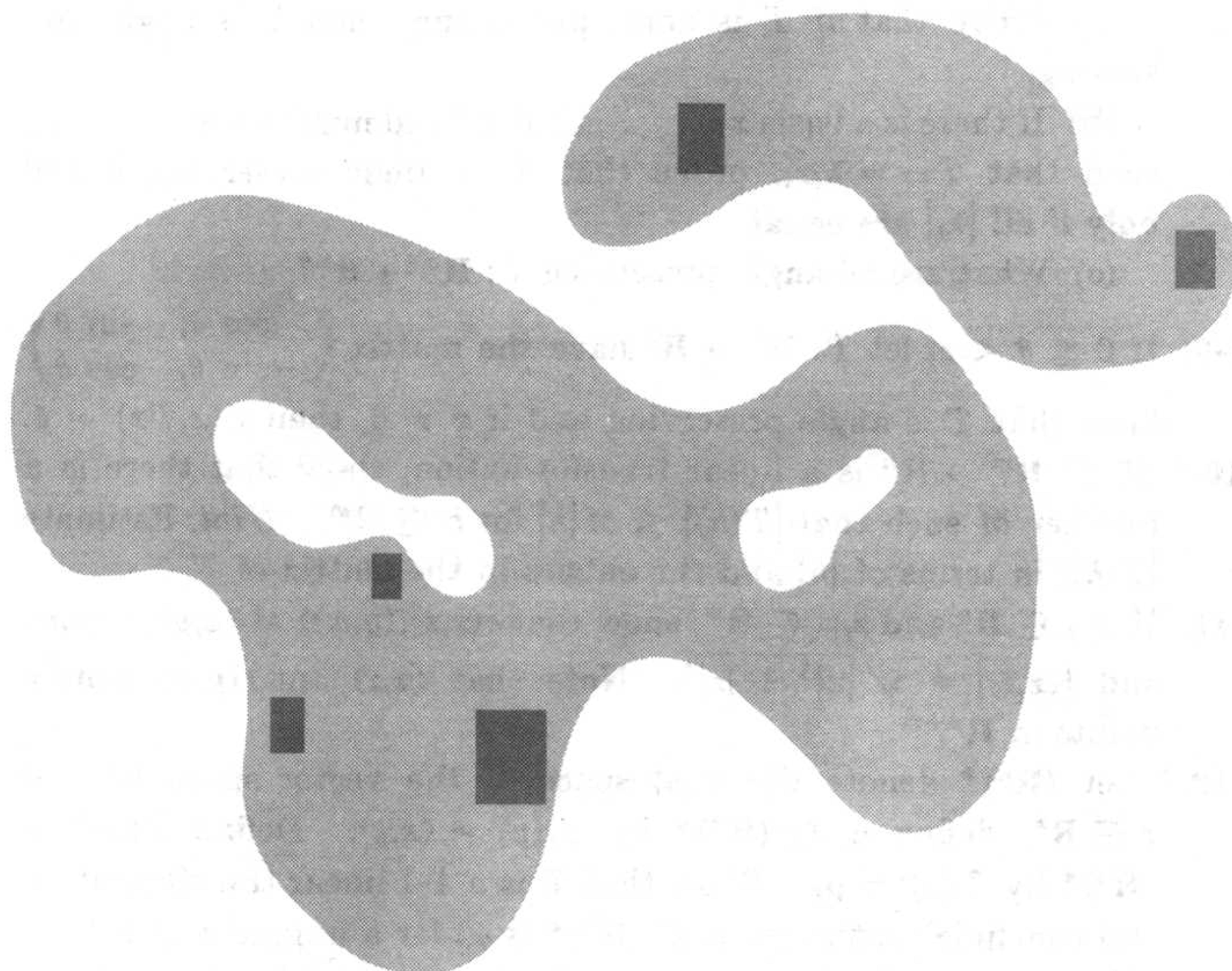


FIGURE 1-1

The reader should supply the proof that a closed rectangle in $\mathbf{R}^n$ is indeed a closed set.

If $A \subset \mathbf{R}^n$ and $x \in \mathbf{R}^n$, then one of three possibilities must hold (Figure 1-2):

1. There is an open rectangle B such that $x \in B \subset A$.
2. There is an open rectangle B such that $x \in B \subset \mathbf{R}^n - A$.
3. If B is any open rectangle with $x \in B$, then B contains points of both A and $\mathbf{R}^n - A$.

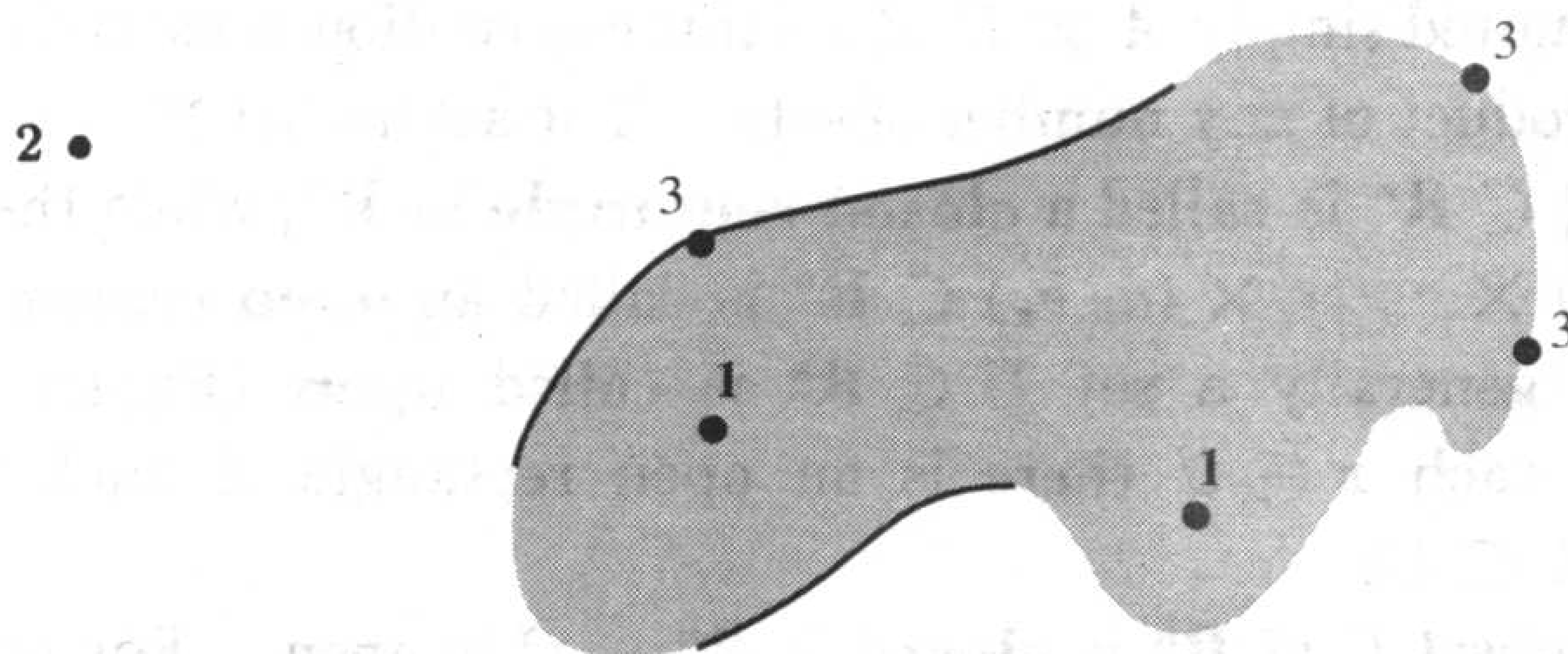


FIGURE 1-2

Those points satisfying (1) constitute the **interior** of A, those satisfying (2) the **exterior** of A, and those satisfying (3) the **boundary** of A. Problems 1-16 to 1-18 show that these terms may sometimes have unexpected meanings.

It is not hard to see that the interior of any set A is open, and the same is true for the exterior of A, which is, in fact, the interior of $\mathbf{R}^n - A$. Thus (Problem 1-14) their union is open, and what remains, the boundary, must be closed.

A collection $\mathcal{O}$ of open sets is an **open cover** of A (or, briefly, **covers** A) if every point $x \in A$ is in some open set in the collection $\mathcal{O}$. For example, if $\mathcal{O}$ is the collection of all open intervals $(a, a + 1)$ for $a \in \mathbf{R}$, then $\mathcal{O}$ is a cover of $\mathbf{R}$. Clearly no finite number of the open sets in $\mathcal{O}$ will cover $\mathbf{R}$ or, for that matter, any unbounded subset of $\mathbf{R}$. A similar situation can also occur for bounded sets. If $\mathcal{O}$ is the collection of all open intervals $(1/n, 1 - 1/n)$ for all integers $n > 1$, then $\mathcal{O}$ is an open cover of $(0,1)$, but again no finite collection of sets in $\mathcal{O}$ will cover $(0,1)$. Although this phenomenon may not appear particularly scandalous, sets for which this state of affairs cannot occur are of such importance that they have received a special designation: a set A is called **compact** if every open cover $\mathcal{O}$ contains a finite subcollection of open sets which also covers A.

A set with only finitely many points is obviously compact and so is the infinite set A which contains 0 and the numbers $1/n$ for all integers n (reason: if $\mathcal{O}$ is a cover, then $0 \in U$ for some open set U in $\mathcal{O}$; there are only finitely many other points of A not in U, each requiring at most one more open set).

Recognizing compact sets is greatly simplified by the following results, of which only the first has any depth (i.e., uses any facts about the real numbers).

1-3 Theorem (Heine-Borel). *The closed interval $[a,b]$ is compact.*

Proof. If $\mathcal{O}$ is an open cover of $[a,b]$, let

$$A = \{x : a \le x \le b \text{ and } [a,x] \text{ is covered by some finite number of open sets in } \mathcal{O}\}.$$

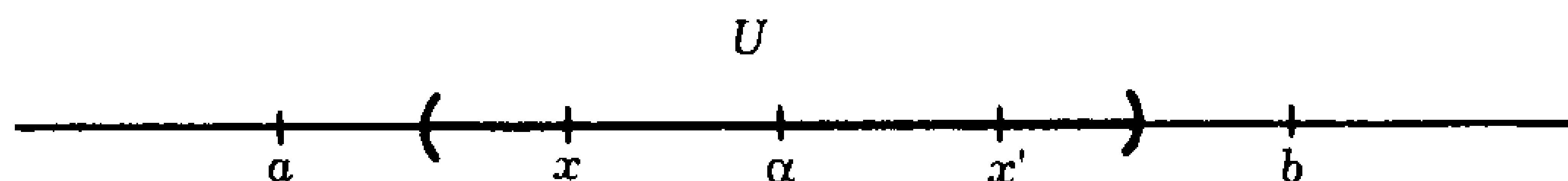


FIGURE 1-3

Note that $a \in A$ and that A is clearly bounded above (by b). We would like to show that $b \in A$. This is done by proving two things about α = least upper bound of A; namely, (1) $\alpha \in A$ and (2) $b = \alpha$.

Since $\mathcal{O}$ is a cover, $\alpha \in U$ for some U in $\mathcal{O}$. Then all points in some interval to the left of α are also in U (see Figure 1-3). Since α is the least upper bound of A, there is an x in this interval such that $x \in A$. Thus $[a,x]$ is covered by some finite number of open sets of $\mathcal{O}$, while $[x,\alpha]$ is covered by the single set U. Hence $[a,\alpha]$ is covered by a finite number of open sets of $\mathcal{O}$, and $\alpha \in A$. This proves (1).

To prove that (2) is true, suppose instead that $\alpha < b$. Then there is a point x' between α and b such that $[\alpha,x'] \subset U$. Since $\alpha \in A$, the interval $[a,\alpha]$ is covered by finitely many open sets of $\mathcal{O}$, while $[\alpha,x']$ is covered by U. Hence $x' \in A$, contradicting the fact that α is an upper bound of A. ▌

If $B \subset \mathbf{R}^m$ is compact and $x \in \mathbf{R}^n$, it is easy to see that $\{x\} \times B \subset \mathbf{R}^{n+m}$ is compact. However, a much stronger assertion can be made.

1-4 Theorem. *If B is compact and $\mathcal{O}$ is an open cover of $\{x\} \times B$, then there is an open set $U \subset \mathbf{R}^n$ containing x such that $U \times B$ is covered by a finite number of sets in $\mathcal{O}$.*

Proof. Since $\{x\} \times B$ is compact, we can assume at the outset that $\mathcal{O}$ is finite, and we need only find the open set U such that $U \times B$ is covered by $\mathcal{O}$.

For each $y \in B$ the point (x,y) is in some open set W in $\mathcal{O}$. Since W is open, we have $(x,y) \in U_y \times V_y \subset W$ for some open rectangle $U_y \times V_y$. The sets V_y cover the compact set B, so a finite number $V_{y_1}, \ldots, V_{y_k}$ also cover B. Let $U = U_{y_1} \cap \cdots \cap U_{y_k}$. Then if $(x',y') \in U \times B$, we have

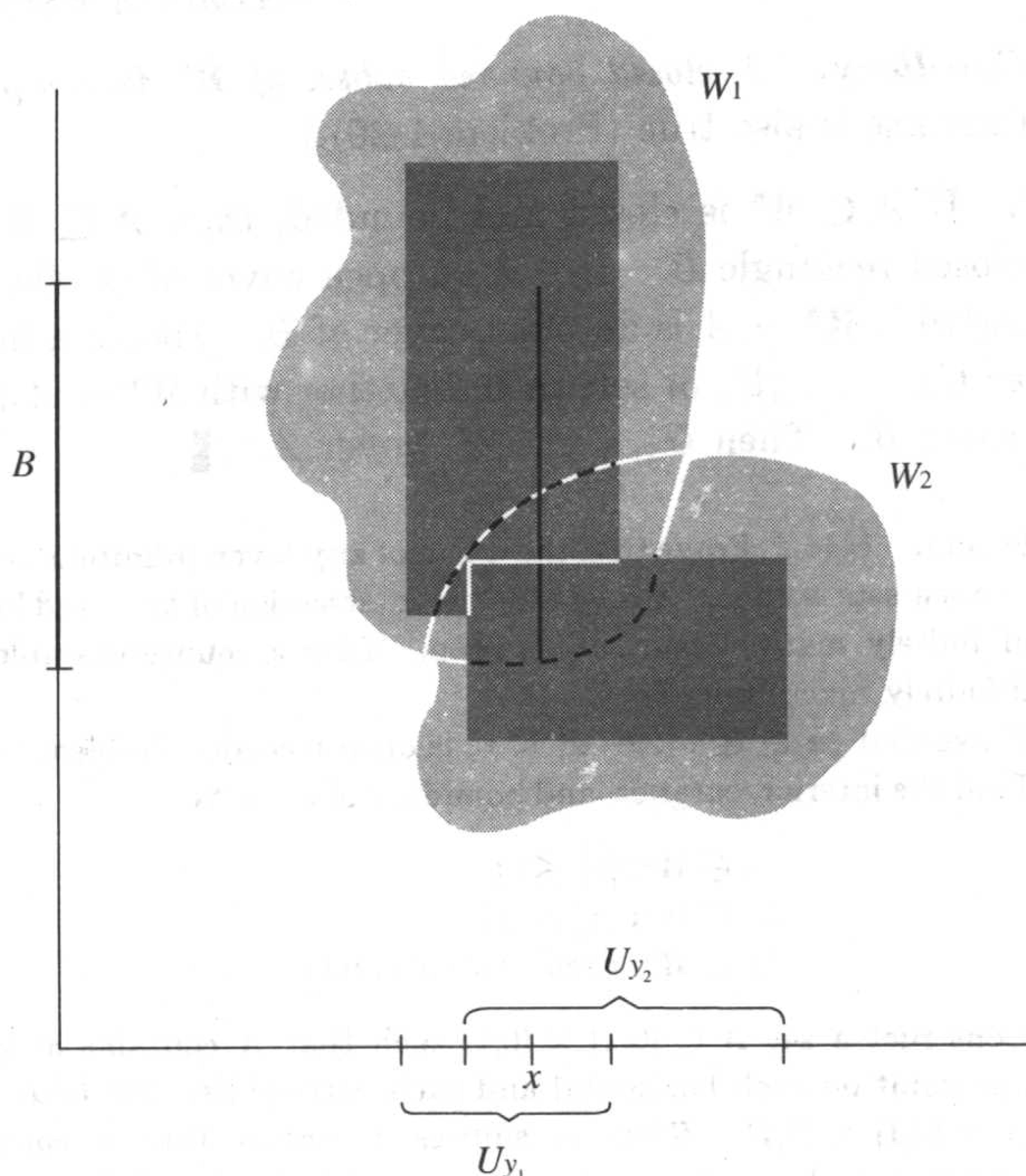


FIGURE 1-4

$y' \in V_{y_i}$ for some i (Figure 1-4), and certainly $x' \in U_{y_i}$. Hence $(x',y') \in U_{y_i} \times V_{y_i}$, which is contained in some W in $\mathcal{O}$. ▮

1-5 Corollary. *If $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^m$ are compact, then $A \times B \subset \mathbf{R}^{n+m}$ is compact.*

Proof. If $\mathcal{O}$ is an open cover of $A \times B$, then $\mathcal{O}$ covers $\{x\} \times B$ for each $x \in A$. By Theorem 1-4 there is an open set U_x containing x such that $U_x \times B$ is covered by finitely many sets in $\mathcal{O}$. Since A is compact, a finite number $U_{x_1}, \ldots, U_{x_n}$ of the U_x cover A. Since finitely many sets in $\mathcal{O}$ cover each $U_{x_i} \times B$, finitely many cover all of $A \times B$. ▮

1-6 Corollary. *$A_1 \times \cdots \times A_k$ is compact if each A_i is. In particular, a closed rectangle in $\mathbf{R}^k$ is compact.*

1-7 Corollary. *A closed bounded subset of* $\mathbf{R}^n$ *is compact.* (The converse is also true (Problem 1-20).)

Proof. If $A \subset \mathbf{R}^n$ is closed and bounded, then $A \subset B$ for some closed rectangle B. If $\mathcal{O}$ is an open cover of A, then $\mathcal{O}$ together with $\mathbf{R}^n - A$ is an open cover of B. Hence a finite number $U_1, \ldots, U_n$ of sets in $\mathcal{O}$, together with $\mathbf{R}^n - A$ perhaps, cover B. Then $U_1, \ldots, U_n$ cover A. ▌

Problems. **1-14.*** Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence of finitely many) open sets is open. Give a counterexample for infinitely many open sets.

1-15. Prove that $\{x \in \mathbf{R}^n : |x - a| < r\}$ is open (see also Problem 1-27).

1-16. Find the interior, exterior, and boundary of the sets

$$\{x \in \mathbf{R}^n : |x| \le 1\}$$
$$\{x \in \mathbf{R}^n : |x| = 1\}$$
$$\{x \in \mathbf{R}^n : \text{each } x^i \text{ is rational}\}.$$

1-17. Construct a set $A \subset [0,1] \times [0,1]$ such that A contains at most one point on each horizontal and each vertical line but boundary $A = [0,1] \times [0,1]$. *Hint:* It suffices to ensure that A contains points in each quarter of the square $[0,1] \times [0,1]$ and also in each sixteenth, etc.

1-18. If $A \subset [0,1]$ is the union of open intervals (a_i,b_i) such that each rational number in $(0,1)$ is contained in some (a_i,b_i), show that boundary $A = [0,1] - A$.

1-19.* If A is a closed set that contains every rational number $r \in [0,1]$, show that $[0,1] \subset A$.

1-20. Prove the converse of Corollary 1-7: A compact subset of $\mathbf{R}^n$ is closed and bounded (see also Problem 1-28).

1-21.* (a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \ge d$ for all $y \in A$.

(b) If A is closed, B is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \ge d$ for all $y \in A$ and $x \in B$. *Hint:* For each $b \in B$ find an open set U containing b such that this relation holds for $x \in U \cap B$.

(c) Give a counterexample in $\mathbf{R}^2$ if A and B are closed but neither is compact.

1-22.* If U is open and $C \subset U$ is compact, show that there is a compact set D such that $C \subset$ interior D and $D \subset U$.

FUNCTIONS AND CONTINUITY

A **function** from $\mathbf{R}^n$ to $\mathbf{R}^m$ (sometimes called a (vector-valued) function of n variables) is a rule which associates to each point in $\mathbf{R}^n$ some point in $\mathbf{R}^m$; the point a function f associates to x is denoted $f(x)$. We write $f: \mathbf{R}^n \to \mathbf{R}^m$ (read "f takes $\mathbf{R}^n$ into $\mathbf{R}^m$" or "f, taking $\mathbf{R}^n$ into $\mathbf{R}^m$," depending on context) to indicate that $f(x) \in \mathbf{R}^m$ is defined for $x \in \mathbf{R}^n$. The notation $f: A \to \mathbf{R}^m$ indicates that $f(x)$ is defined only for x in the set A, which is called the **domain** of f. If $B \subset A$, we define $f(B)$ as the set of all $f(x)$ for $x \in B$, and if $C \subset \mathbf{R}^m$ we define $f^{-1}(C) = \{x \in A : f(x) \in C\}$. The notation $f: A \to B$ indicates that $f(A) \subset B$.

A convenient representation of a function $f: \mathbf{R}^2 \to \mathbf{R}$ may be obtained by drawing a picture of its graph, the set of all 3-tuples of the form $(x,y,f(x,y))$, which is actually a figure in 3-space (see, e.g., Figures 2-1 and 2-2 of Chapter 2).

If $f,g: \mathbf{R}^n \to \mathbf{R}$, the functions $f + g, f - g, f \cdot g$, and f/g are defined precisely as in the one-variable case. If $f: A \to \mathbf{R}^m$ and $g: B \to \mathbf{R}^p$, where $B \subset \mathbf{R}^m$, then the **composition** $g \circ f$ is defined by $g \circ f(x) = g(f(x))$; the domain of $g \circ f$ is $A \cap f^{-1}(B)$. If $f: A \to \mathbf{R}^m$ is 1-1, that is, if $f(x) \neq f(y)$ when $x \neq y$, we define $f^{-1}: f(A) \to \mathbf{R}^n$ by the requirement that $f^{-1}(z)$ is the unique $x \in A$ with $f(x) = z$.

A function $f: A \to \mathbf{R}^m$ determines m **component functions** $f^1, \ldots, f^m: A \to \mathbf{R}$ by $f(x) = (f^1(x), \ldots, f^m(x))$. If conversely, m functions $g_1, \ldots, g_m: A \to \mathbf{R}$ are given, there is a unique function $f: A \to \mathbf{R}^m$ such that $f^i = g_i$, namely $f(x) = (g_1(x), \ldots, g_m(x))$. This function f will be denoted $(g_1, \ldots, g_m)$, so that we always have $f = (f^1, \ldots, f^m)$. If $\pi: \mathbf{R}^n \to \mathbf{R}^n$ is the identity function, $\pi(x) = x$, then $\pi^i(x) = x^i$; the function π^i is called the ith **projection function.**

The notation $\lim_{x \to a} f(x) = b$ means, as in the one-variable case, that we can get $f(x)$ as close to b as desired, by choosing x sufficiently close to, but not equal to, a. In mathematical terms this means that for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all x in the domain of f which

satisfy $0 < |x - a| < \delta$. A function $f\colon A \to \mathbf{R}^m$ is called **continuous** at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$, and f is simply called continuous if it is continuous at each $a \in A$. One of the pleasant surprises about the concept of continuity is that it can be defined without using limits. It follows from the next theorem that $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is continuous if and only if $f^{-1}(U)$ is open whenever $U \subset \mathbf{R}^m$ is open; if the domain of f is not all of $\mathbf{R}^n$, a slightly more complicated condition is needed.

1-8 Theorem. *If $A \subset \mathbf{R}^n$, a function $f\colon A \to \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is an open rectangle B with $f(a) \in B \subset U$. Since f is continuous at a, we can ensure that $f(x) \in B$, provided we choose x in some sufficiently small rectangle C containing a. Do this for each $a \in f^{-1}(U)$ and let V be the union of all such C. Clearly $f^{-1}(U) = V \cap A$. The converse is similar and is left to the reader. ▌

The following consequence of Theorem 1-8 is of great importance.

1-9 Theorem. ***If $f\colon A \to \mathbf{R}^m$ is continuous, where $A \subset \mathbf{R}^n$, and A is compact, then $f(A) \subset \mathbf{R}^m$ is compact.***

Proof. Let $\mathcal{O}$ be an open cover of $f(A)$. For each open set U in $\mathcal{O}$ there is an open set V_U such that $f^{-1}(U) = V_U \cap A$. The collection of all V_U is an open cover of A. Since A is compact, a finite number $V_{U_1}, \ldots, V_{U_n}$ cover A. Then $U_1, \ldots, U_n$ cover $f(A)$. ▌

If $f\colon A \to \mathbf{R}$ is bounded, the extent to which f fails to be continuous at $a \in A$ can be measured in a precise way. For $\delta > 0$ let

$$M(a,f,\delta) = \sup\{f(x) : x \in A \text{ and } |x - a| < \delta\},$$
$$m(a,f,\delta) = \inf\{f(x) : x \in A \text{ and } |x - a| < \delta\}.$$

The **oscillation** $o(f,a)$ of f at a is defined by $o(f,a) = \lim_{\delta \to 0}[M(a,f,\delta) - m(a,f,\delta)]$. This limit always exists, since $M(a,f,\delta) - m(a,f,\delta)$ decreases as δ decreases. There are two important facts about $o(f,a)$.

1-10 Theorem. *The bounded function f is continuous at a if and only if $o(f,a) = 0$.*

Proof. Let f be continuous at a. For every number $\varepsilon > 0$ we can choose a number $\delta > 0$ so that $|f(x) - f(a)| < \varepsilon$ for all $x \in A$ with $|x - a| < \delta$; thus $M(a,f,\delta) - m(a,f,\delta) \leq 2\varepsilon$. Since this is true for every ε, we have $o(f,a) = 0$. The converse is similar and is left to the reader. ▮

1-11 Theorem. *Let $A \subset \mathbf{R}^n$ be closed. If $f: A \to \mathbf{R}$ is any bounded function, and $\varepsilon > 0$, then $\{x \in A: o(f,x) \geq \varepsilon\}$ is closed.*

Proof. Let $B = \{x \in A: o(f,x) \geq \varepsilon\}$. We wish to show that $\mathbf{R}^n - B$ is open. If $x \in \mathbf{R}^n - B$, then either $x \notin A$ or else $x \in A$ and $o(f,x) < \varepsilon$. In the first case, since A is closed, there is an open rectangle C containing x such that $C \subset \mathbf{R}^n - A \subset \mathbf{R}^n - B$. In the second case there is a $\delta > 0$ such that $M(x,f,\delta) - m(x,f,\delta) < \varepsilon$. Let C be an open rectangle containing x such that $|x - y| < \delta$ for all $y \in C$. Then if $y \in C$ there is a δ_1 such that $|x - z| < \delta$ for all z satisfying $|z - y| < \delta_1$. Thus $M(y,f,\delta_1) - m(y,f,\delta_1) < \varepsilon$, and consequently $o(y,f) < \varepsilon$. Therefore $C \subset \mathbf{R}^n - B$. ▮

Problems. **1-23.** If $f: A \to \mathbf{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$.

1-24. Prove that $f: A \to \mathbf{R}^m$ is continuous at a if and only if each f^i is.

1-25. Prove that a linear transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is continuous. *Hint:* Use Problem 1-10.

1-26. Let $A = \{(x,y) \in \mathbf{R}^2: x > 0 \text{ and } 0 < y < x^2\}$.

(a) Show that every straight line through (0,0) contains an interval around (0,0) which is in $\mathbf{R}^2 - A$.

(b) Define $f: \mathbf{R}^2 \to \mathbf{R}$ by $f(x) = 0$ if $x \notin A$ and $f(x) = 1$ if $x \in A$. For $h \in \mathbf{R}^2$ define $g_h: \mathbf{R} \to \mathbf{R}$ by $g_h(t) = f(th)$. Show that each g_h is continuous at 0, but f is not continuous at (0,0).

1-27. Prove that $\{x \in \mathbf{R}^n\colon |x - a| < r\}$ is open by considering the function $f\colon \mathbf{R}^n \to \mathbf{R}$ with $f(x) = |x - a|$.

1-28. If $A \subset \mathbf{R}^n$ is not closed, show that there is a continuous function $f\colon A \to \mathbf{R}$ which is unbounded. *Hint:* If $x \in \mathbf{R}^n - A$ but $x \notin$ interior $(\mathbf{R}^n - A)$, let $f(y) = 1/|y - x|$.

1-29. If A is compact, prove that every continuous function $f\colon A \to \mathbf{R}$ takes on a maximum and a minimum value.

1-30. Let $f\colon [a,b] \to \mathbf{R}$ be an increasing function. If $x_1, \ldots, x_n \in [a,b]$ are distinct, show that $\sum_{i=1}^{n} o(f,x_i) < f(b) - f(a)$.

2

Differentiation

BASIC DEFINITIONS

Recall that a function $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable at $a \in \mathbf{R}$ if there is a number $f'(a)$ such that

$$(1)\ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

This equation certainly makes no sense in the general case of a function $f\colon \mathbf{R}^n \to \mathbf{R}^m$, but can be reformulated in a way that does. If $\lambda\colon \mathbf{R} \to \mathbf{R}$ is the linear transformation defined by $\lambda(h) = f'(a) \cdot h$, then equation (1) is equivalent to

$$(2)\ \lim_{h \to 0} \frac{f(a+h) - f(a) - \lambda(h)}{h} = 0.$$

Equation (2) is often interpreted as saying that $\lambda + f(a)$ is a good approximation to f at a (see Problem 2-9). Henceforth we focus our attention on the linear transformation λ and reformulate the definition of differentiability as follows.

A function $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable at $a \in \mathbf{R}$ if there is a linear transformation $\lambda\colon \mathbf{R} \to \mathbf{R}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \lambda(h)}{h} = 0.$$

In this form the definition has a simple generalization to higher dimensions:

A function $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is **differentiable** at $a \in \mathbf{R}^n$ if there is a linear transformation $\lambda\colon \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Note that h is a point of $\mathbf{R}^n$ and $f(a+h) - f(a) - \lambda(h)$ a point of $\mathbf{R}^m$, so the norm signs are essential. The linear transformation λ is denoted $Df(a)$ and called the **derivative** of f at a. The justification for the phrase "*the* linear transformation λ" is

2-1 ***Theorem.*** *If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$ there is a* unique *linear transformation $\lambda\colon \mathbf{R}^n \to \mathbf{R}^m$ such that*

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Proof. Suppose $\mu\colon \mathbf{R}^n \to \mathbf{R}^m$ satisfies

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} = 0.$$

If $d(h) = f(a+h) - f(a)$, then

$$\begin{aligned}
\lim_{h \to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} &= \lim_{h \to 0} \frac{|\lambda(h) - d(h) + d(h) - \mu(h)|}{|h|} \\
&\le \lim_{h \to 0} \frac{|\lambda(h) - d(h)|}{|h|} + \lim_{h \to 0} \frac{|d(h) - \mu(h)|}{|h|} \\
&= 0.
\end{aligned}$$

If $x \in \mathbf{R}^n$, then $tx \to 0$ as $t \to 0$. Hence for $x \neq 0$ we have

$$0 = \lim_{t \to 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} = \frac{|\lambda(x) - \mu(x)|}{|x|}.$$

Therefore $\lambda(x) = \mu(x)$. ▮

We shall later discover a simple way of finding $Df(a)$. For the moment let us consider the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x,y) = \sin x$. Then $Df(a,b) = \lambda$ satisfies $\lambda(x,y) = (\cos a) \cdot x$. To prove this, note that

$$\lim_{(h,k)\to 0} \frac{|f(a+h, b+k) - f(a,b) - \lambda(h,k)|}{|(h,k)|} = \lim_{(h,k)\to 0} \frac{|\sin(a+h) - \sin a - (\cos a)\cdot h|}{|(h,k)|}.$$

Since $\sin'(a) = \cos a$, we have

$$\lim_{h\to 0} \frac{|\sin(a+h) - \sin a - (\cos a)\cdot h|}{|h|} = 0.$$

Since $|(h,k)| \geq |h|$, it is also true that

$$\lim_{h\to 0} \frac{|\sin(a+h) - \sin a - (\cos a)\cdot h|}{|(h,k)|} = 0.$$

It is often convenient to consider the matrix of $Df(a)\colon \mathbf{R}^n \to \mathbf{R}^m$ with respect to the usual bases of $\mathbf{R}^n$ and $\mathbf{R}^m$. This $m \times n$ matrix is called the **Jacobian matrix** of f at a, and denoted $f'(a)$. If $f(x,y) = \sin x$, then $f'(a,b) = (\cos a, 0)$. If $f\colon \mathbf{R} \to \mathbf{R}$, then $f'(a)$ is a 1×1 matrix whose single entry is the number which is denoted $f'(a)$ in elementary calculus.

The definition of $Df(a)$ could be made if f were defined only in some open set containing a. Considering only functions defined on $\mathbf{R}^n$ streamlines the statement of theorems and produces no real loss of generality. It is convenient to define a function $f\colon \mathbf{R}^n \to \mathbf{R}^m$ to be **differentiable on** A if f is differentiable at a for each $a \in A$. If $f\colon A \to \mathbf{R}^m$, then f is called differentiable if f can be extended to a differentiable function on some open set containing A.

Problems. **2-1.*** Prove that if $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a. *Hint:* Use Problem 1-10.

2-2. A function $f\colon \mathbf{R}^2 \to \mathbf{R}$ is **independent of the second variable** if for each $x \in \mathbf{R}$ we have $f(x,y_1) = f(x,y_2)$ for all $y_1,y_2 \in \mathbf{R}$. Show that f is independent of the second variable if and only if there is a function $g\colon \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g(x)$. What is $f'(a,b)$ in terms of g'?

2-3. Define when a function $f: \mathbf{R}^2 \to \mathbf{R}$ is independent of the first variable and find $f'(a,b)$ for such f. Which functions are independent of the first variable and also of the second variable?

2-4. Let g be a continuous real-valued function on the unit circle $\{x \in \mathbf{R}^2: |x| = 1\}$ such that $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$. Define $f: \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} |x| \cdot g\left(\dfrac{x}{|x|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) If $x \in \mathbf{R}^2$ and $h: \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

(b) Show that f is not differentiable at $(0,0)$ unless $g = 0$. *Hint:* First show that $Df(0,0)$ would have to be 0 by considering (h,k) with $k = 0$ and then with $h = 0$.

2-5. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2 + y^2}} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

Show that f is a function of the kind considered in Problem 2-4, so that f is not differentiable at $(0,0)$.

2-6. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x,y) = \sqrt{|xy|}$. Show that f is not differentiable at $(0,0)$.

2-7. Let $f: \mathbf{R}^n \to \mathbf{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.

2-8. Let $f: \mathbf{R} \to \mathbf{R}^2$. Prove that f is differentiable at $a \in \mathbf{R}$ if and only if f^1 and f^2 are, and that in this case

$$f'(a) = \begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \end{pmatrix}.$$

2-9. Two functions $f,g: \mathbf{R} \to \mathbf{R}$ are **equal up to nth order** at a if

$$\lim_{h \to 0} \frac{f(a+h) - g(a+h)}{h^n} = 0.$$

(a) Show that f is differentiable at a if and only if there is a function g of the form $g(x) = a_0 + a_1(x - a)$ such that f and g are equal up to first order at a.

(b) If $f'(a), \ldots, f^{(n)}(a)$ exist, show that f and the function g defined by

$$g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i$$

are equal up to nth order at a. *Hint:* The limit

$$\lim_{x \to a} \frac{f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n}$$

may be evaluated by L'Hospital's rule.

BASIC THEOREMS

2-2 Theorem (Chain Rule). *If $f: \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a, and $g: \mathbf{R}^m \to \mathbf{R}^p$ is differentiable at $f(a)$, then the composition $g \circ f: \mathbf{R}^n \to \mathbf{R}^p$ is differentiable at a, and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a).$$

Remark. This equation can be written

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

If $m = n = p = 1$, we obtain the old chain rule.

Proof. Let $b = f(a)$, let $\lambda = Df(a)$, and let $\mu = Dg(f(a))$. If we define

$$\begin{aligned}
&(1)\ \ \varphi(x) = f(x) - f(a) - \lambda(x-a),\\
&(2)\ \ \psi(y) = g(y) - g(b) - \mu(y-b),\\
&(3)\ \ \rho(x) = g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x-a),
\end{aligned}$$

then

$$(4)\ \ \lim_{x \to a} \frac{|\varphi(x)|}{|x-a|} = 0,$$

$$(5)\ \ \lim_{y \to b} \frac{|\psi(y)|}{|y-b|} = 0,$$

and we must show that

$$\lim_{x \to a} \frac{|\rho(x)|}{|x-a|} = 0.$$

Now

$$\begin{aligned}
\rho(x) &= g(f(x)) - g(b) - \mu(\lambda(x-a))\\
&= g(f(x)) - g(b) - \mu(f(x) - f(a) - \varphi(x)) \qquad \text{by (1)}\\
&= [g(f(x)) - g(b) - \mu(f(x) - f(a))] + \mu(\varphi(x))\\
&= \psi(f(x)) + \mu(\varphi(x)) \qquad \text{by (2)}.
\end{aligned}$$

Thus we must prove

$$(6)\ \lim_{x \to a} \frac{|\psi(f(x))|}{|x - a|} = 0,$$

$$(7)\ \lim_{x \to a} \frac{|\mu(\varphi(x))|}{|x - a|} = 0.$$

Equation (7) follows easily from (4) and Problem 1-10. If $\varepsilon > 0$ it follows from (5) that for some $\delta > 0$ we have

$$|\psi(f(x))| < \varepsilon|f(x) - b| \qquad \text{if } |f(x) - b| < \delta,$$

which is true if $|x - a| < \delta_1$, for a suitable δ_1. Then

$$\begin{aligned} |\psi(f(x))| &< \varepsilon|f(x) - b| \\ &= \varepsilon|\varphi(x) + \lambda(x - a)| \\ &\leq \varepsilon|\varphi(x)| + \varepsilon M|x - a| \end{aligned}$$

for some M, by Problem 1-10. Equation (6) now follows easily. ▌

2-3 *Theorem*

(1) *If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is a constant function (that is, if for some $y \in \mathbf{R}^m$ we have $f(x) = y$ for all $x \in \mathbf{R}^n$), then*

$$Df(a) = 0.$$

(2) *If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then*

$$Df(a) = f.$$

(3) *If $f\colon \mathbf{R}^n \to \mathbf{R}^m$, then f is differentiable at $a \in \mathbf{R}^n$ if and only if each f^i is, and*

$$Df(a) = (Df^1(a), \ldots, Df^m(a)).$$

Thus $f'(a)$ is the $m \times n$ matrix whose ith row is $(f^i)'(a)$.

(4) *If $s\colon \mathbf{R}^2 \to \mathbf{R}$ is defined by $s(x,y) = x + y$, then*

$$Ds(a,b) = s.$$

(5) *If $p\colon \mathbf{R}^2 \to \mathbf{R}$ is defined by $p(x,y) = x \cdot y$, then*

$$Dp(a,b)(x,y) = bx + ay.$$

Thus $p'(a,b) = (b,a)$.

Proof

(1) $$\lim_{h\to 0} \frac{|f(a+h) - f(a) - 0|}{|h|} = \lim_{h\to 0} \frac{|y - y - 0|}{|h|} = 0.$$

(2) $$\lim_{h\to 0} \frac{|f(a+h) - f(a) - f(h)|}{|h|}$$
$$= \lim_{h\to 0} \frac{|f(a) + f(h) - f(a) - f(h)|}{|h|} = 0.$$

(3) If each f^i is differentiable at a and

$$\lambda = (Df^1(a), \ldots, Df^m(a)),$$

then

$$f(a+h) - f(a) - \lambda(h)$$
$$= (f^1(a+h) - f^1(a) - Df^1(a)(h), \ldots,$$
$$f^m(a+h) + f^m(a) - Df^m(a)(h)).$$

Therefore

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|}$$
$$\leq \lim_{h\to 0} \sum_{i=1}^{m} \frac{|f^i(a+h) - f^i(a) - Df^i(a)(h)|}{|h|} = 0.$$

If, on the other hand, f is differentiable at a, then $f^i = \pi^i \circ f$ is differentiable at a by (2) and Theorem 2-2.

(4) follows from (2).

(5) Let $\lambda(x,y) = bx + ay$. Then

$$\lim_{(h,k)\to 0} \frac{|p(a+h, b+k) - p(a,b) - \lambda(h,k)|}{|(h,k)|}$$
$$= \lim_{(h,k)\to 0} \frac{|hk|}{|(h,k)|}.$$

Now

$$|hk| \leq \begin{cases} |h|^2 & \text{if } |k| \leq |h|, \\ |k|^2 & \text{if } |h| \leq |k|. \end{cases}$$

Hence $|hk| \leq |h|^2 + |k|^2$. Therefore

$$\frac{|hk|}{|(h,k)|} \leq \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2},$$

so

$$\lim_{(h,k)\to 0} \frac{|hk|}{|(h,k)|} = 0. \quad \blacksquare$$

2-4 Corollary. *If $f,g\colon \mathbf{R}^n \to \mathbf{R}$ are differentiable at a, then*

$$\begin{aligned} D(f+g)(a) &= Df(a) + Dg(a), \\ D(f \cdot g)(a) &= g(a)Df(a) + f(a)Dg(a). \end{aligned}$$

If, moreover, $g(a) \neq 0$, then

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}.$$

Proof. We will prove the first equation and leave the others to the reader. Since $f + g = s \circ (f,g)$, we have

$$\begin{aligned} D(f+g)(a) &= Ds(f(a),g(a)) \circ D(f,g)(a) \\ &= s \circ (Df(a),Dg(a)) \\ &= Df(a) + Dg(a). \quad \blacksquare \end{aligned}$$

We are now assured of the differentiability of those functions $f\colon \mathbf{R}^n \to \mathbf{R}^m$, whose component functions are obtained by addition, multiplication, division, and composition, from the functions π^i (which are linear transformations) and the functions which we can already differentiate by elementary calculus. Finding $Df(x)$ or $f'(x)$, however, may be a fairly formidable task. For example, let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x,y) = \sin(xy^2)$. Since $f = \sin \circ (\pi^1 \cdot [\pi^2]^2)$, we have

$$\begin{aligned} f'(a,b) &= \sin'(ab^2) \cdot [b^2(\pi^1)'(a,b) + a([\pi^2]^2)'(a,b)] \\ &= \sin'(ab^2) \cdot [b^2(\pi^1)'(a,b) + 2ab(\pi^2)'(a,b)] \\ &= (\cos(ab^2)) \cdot [b^2(1,0) + 2ab(0,1)] \\ &= (b^2\cos(ab^2),\, 2ab\cos(ab^2)). \end{aligned}$$

Fortunately, we will soon discover a much simpler method of computing f'.

Problems. **2-10.** Use the theorems of this section to find f' for the following:

(a) $f(x,y,z) = x^y$.
(b) $f(x,y,z) = (x^y,z)$.
(c) $f(x,y) = \sin(x \sin y)$.
(d) $f(x,y,z) = \sin(x \sin(y \sin z))$.
(e) $f(x,y,z) = x^{y^z}$.
(f) $f(x,y,z) = x^{y+z}$.
(g) $f(x,y,z) = (x + y)^z$.
(h) $f(x,y) = \sin(xy)$.
(i) $f(x,y) = [\sin(xy)]^{\cos 3}$.
(j) $f(x,y) = (\sin(xy), \sin(x \sin y), x^y)$.

2-11. Find f' for the following (where $g\colon \mathbf{R} \to \mathbf{R}$ is continuous):
(a) $f(x,y) = \int_a^{x+y} g$.
(b) $f(x,y) = \int_a^{x \cdot y} g$.
(c) $f(x,y,z) = \int_{xy}^{\sin(x \sin(y \sin z))} g$.

2-12. A function $f\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^p$ is **bilinear** if for $x,x_1,x_2 \in \mathbf{R}^n$, $y,y_1,y_2 \in \mathbf{R}^m$, and $a \in \mathbf{R}$ we have

$$f(ax,y) = af(x,y) = f(x,ay),$$
$$f(x_1 + x_2,y) = f(x_1,y) + f(x_2,y),$$
$$f(x,y_1 + y_2) = f(x,y_1) + f(x,y_2).$$

(a) Prove that if f is bilinear, then

$$\lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0.$$

(b) Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.
(c) Show that the formula for $Dp(a,b)$ in Theorem 2-3 is a special case of (b).

2-13. Define $IP\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by $IP(x,y) = \langle x,y \rangle$.
(a) Find $D(IP)(a,b)$ and $(IP)'(a,b)$.
(b) If $f,g\colon \mathbf{R} \to \mathbf{R}^n$ are differentiable and $h\colon \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = \langle f(t),g(t) \rangle$, show that

$$h'(a) = \langle f'(a)^{\mathrm{T}},g(a) \rangle + \langle f(a),g'(a)^{\mathrm{T}} \rangle.$$

(Note that $f'(a)$ is an $n \times 1$ matrix; its transpose $f'(a)^{\mathrm{T}}$ is a $1 \times n$ matrix, which we consider as a member of $\mathbf{R}^n$.)
(c) If $f\colon \mathbf{R} \to \mathbf{R}^n$ is differentiable and $|f(t)| = 1$ for all t, show that $\langle f'(t)^{\mathrm{T}},f(t) \rangle = 0$.
(d) Exhibit a differentiable function $f\colon \mathbf{R} \to \mathbf{R}$ such that the function $|f|$ defined by $|f|(t) = |f(t)|$ is not differentiable.

2-14. Let E_i, $i = 1, \ldots, k$ be Euclidean spaces of various dimensions. A function $f\colon E_1 \times \cdots \times E_k \to \mathbf{R}^p$ is called **multilinear** if for each choice of $x_j \in E_j$, $j \neq i$ the function $g\colon E_i \to \mathbf{R}^p$ defined by $g(x) = f(x_1, \ldots, x_{i-1},x,x_{i+1}, \ldots, x_k)$ is a linear transformation.

(a) If f is multilinear and $i \neq j$, show that for $h = (h_1, \ldots, h_k)$, with $h_l \in E_l$, we have

$$\lim_{h \to 0} \frac{|f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)|}{|h|} = 0.$$

Hint: If $g(x,y) = f(a_1, \ldots, x, \ldots, y, \ldots, a_k)$, then g is bilinear.

(b) Prove that

$$Df(a_1, \ldots, a_k)(x_1, \ldots, x_k) = \sum_{i=1}^{k} f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k).$$

2-15. Regard an $n \times n$ matrix as a point in the n-fold product $\mathbf{R}^n \times \cdots \times \mathbf{R}^n$ by considering each row as a member of $\mathbf{R}^n$.

(a) Prove that $\det: \mathbf{R}^n \times \cdots \times \mathbf{R}^n \to \mathbf{R}$ is differentiable and

$$D(\det)(a_1, \ldots, a_n)(x_1, \ldots, x_n) = \sum_{i=1}^{n} \det \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ x_i \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}.$$

(b) If $a_{ij}: \mathbf{R} \to \mathbf{R}$ are differentiable and $f(t) = \det(a_{ij}(t))$, show that

$$f'(t) = \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(t), & \ldots, & a_{1n}(t) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{j1}'(t), & \ldots, & a_{jn}'(t) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1}(t), & \ldots, & a_{nn}(t) \end{pmatrix}.$$

(c) If $\det(a_{ij}(t)) \neq 0$ for all t and $b_1, \ldots, b_n: \mathbf{R} \to \mathbf{R}$ are differentiable, let $s_1, \ldots, s_n: \mathbf{R} \to \mathbf{R}$ be the functions such that $s_1(t), \ldots, s_n(t)$ are the solutions of the equations

$$\sum_{j=1}^{n} a_{ji}(t) s_j(t) = b_i(t) \qquad i = 1, \ldots, n.$$

Show that s_i is differentiable and find $s_i'(t)$.

2-16. Suppose $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is differentiable and has a differentiable inverse $f^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$. Show that $(f^{-1})'(a) = [f'(f^{-1}(a))]^{-1}$. *Hint:* $f \circ f^{-1}(x) = x$.

PARTIAL DERIVATIVES

We begin the attack on the problem of finding derivatives "one variable at a time." If $f\colon \mathbf{R}^n \to \mathbf{R}$ and $a \in \mathbf{R}^n$, the limit

$$\lim_{h\to 0} \frac{f(a^1, \ldots, a^i + h, \ldots, a^n) - f(a^1, \ldots, a^n)}{h},$$

if it exists, is denoted $D_if(a)$, and called the ith **partial derivative** of f at a. It is important to note that $D_if(a)$ is the ordinary derivative of a certain function; in fact, if $g(x) = f(a^1, \ldots, x, \ldots, a^n)$, then $D_if(a) = g'(a^i)$. This means that $D_if(a)$ is the slope of the tangent line at $(a,f(a))$ to the curve obtained by intersecting the graph of f with the plane $x^j = a^j$, $j \neq i$ (Figure 2-1). It also means that computation of $D_if(a)$ is a problem we can already solve. If $f(x^1, \ldots, x^n)$ is

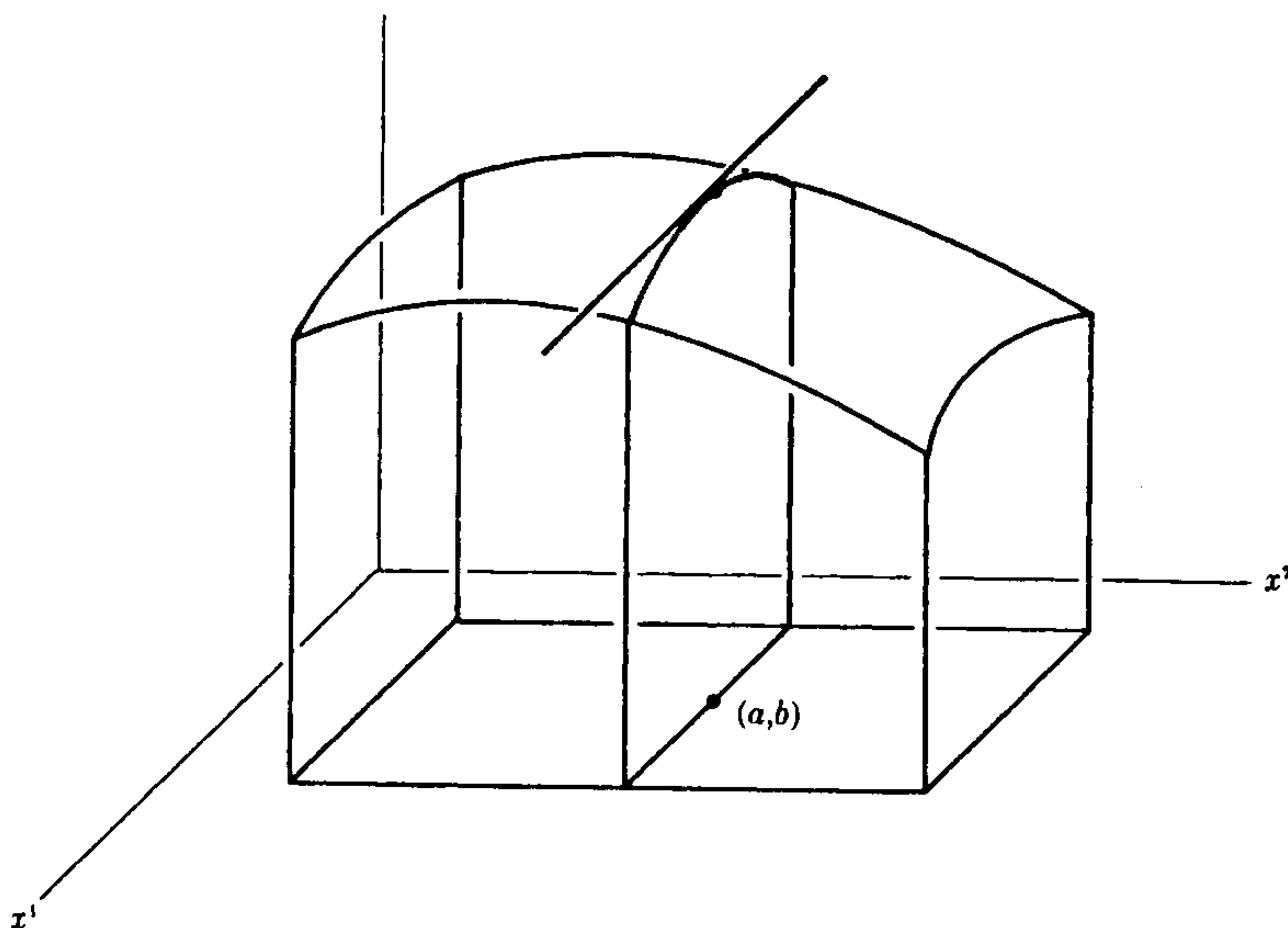


FIGURE 2-1

given by some formula involving $x^1, \ldots, x^n$, then we find $D_if(x^1, \ldots, x^n)$ by differentiating the function whose value at x^i is given by the formula when all x^j, for $j \neq i$, are thought of as constants. For example, if $f(x,y) = \sin(xy^2)$, then $D_1f(x,y) = y^2\cos(xy^2)$ and $D_2f(x,y) = 2xy\cos(xy^2)$. If, instead, $f(x,y) = x^y$, then $D_1f(x,y) = yx^{y-1}$ and $D_2f(x,y) = x^y \log x$.

With a little practice (e.g., the problems at the end of this section) you should acquire as great a facility for computing D_if as you already have for computing ordinary derivatives.

If $D_if(x)$ exists for all $x \in \mathbf{R}^n$, we obtain a function D_if: $\mathbf{R}^n \to \mathbf{R}$. The jth partial derivative of this function at x, that is, $D_j(D_if)(x)$, is often denoted $D_{i,j}f(x)$. Note that this notation reverses the order of i and j. As a matter of fact, the order is usually irrelevant, since most functions (an exception is given in the problems) satisfy $D_{i,j}f = D_{j,i}f$. There are various delicate theorems ensuring this equality; the following theorem is quite adequate. We state it here but postpone the proof until later (Problem 3-28).

2-5 *Theorem.* *If $D_{i,j}f$ and $D_{j,i}f$ are continuous in an open set containing a, then*

$$D_{i,j}f(a) = D_{j,i}f(a).$$

The function $D_{i,j}f$ is called a **second-order (mixed) partial derivative** of f. Higher-order (mixed) partial derivatives are defined in the obvious way. Clearly Theorem 2-5 can be used to prove the equality of higher-order mixed partial derivatives under appropriate conditions. The order of $i_1, \ldots, i_k$ is completely immaterial in $D_{i_1,\ldots,i_k}f$ if f has continuous partial derivatives of all orders. A function with this property is called a C^∞ function. In later chapters it will frequently be convenient to restrict our attention to C^∞ functions.

Partial derivatives will be used in the next section to find derivatives. They also have another important use—finding maxima and minima of functions.

2-6 Theorem. *Let $A \subset \mathbf{R}^n$. If the maximum (or minimum) of $f\colon A \to \mathbf{R}$ occurs at a point a in the interior of A and $D_if(a)$ exists, then $D_if(a) = 0$.*

Proof. Let $g_i(x) = f(a^1, \ldots, x, \ldots, a^n)$. Clearly g_i has a maximum (or minimum) at a^i, and g_i is defined in an open interval containing a^i. Hence $0 = g_i'(a^i) = D_if(a)$. ▌

The reader is reminded that the converse of Theorem 2-6 is false even if $n = 1$ (if $f\colon \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^3$, then $f'(0) = 0$, but 0 is not even a local maximum or minimum). If $n > 1$, the converse of Theorem 2-6 may fail to be true in a rather spectacular way. Suppose, for example, that $f\colon \mathbf{R}^2 \to \mathbf{R}$ is defined by $f(x,y) = x^2 - y^2$ (Figure 2-2). Then $D_1f(0,0) = 0$ because g_1 has a minimum at 0, while $D_2f(0,0) = 0$ because g_2 has a maximum at 0. Clearly $(0,0)$ is neither a relative maximum nor a relative minimum.

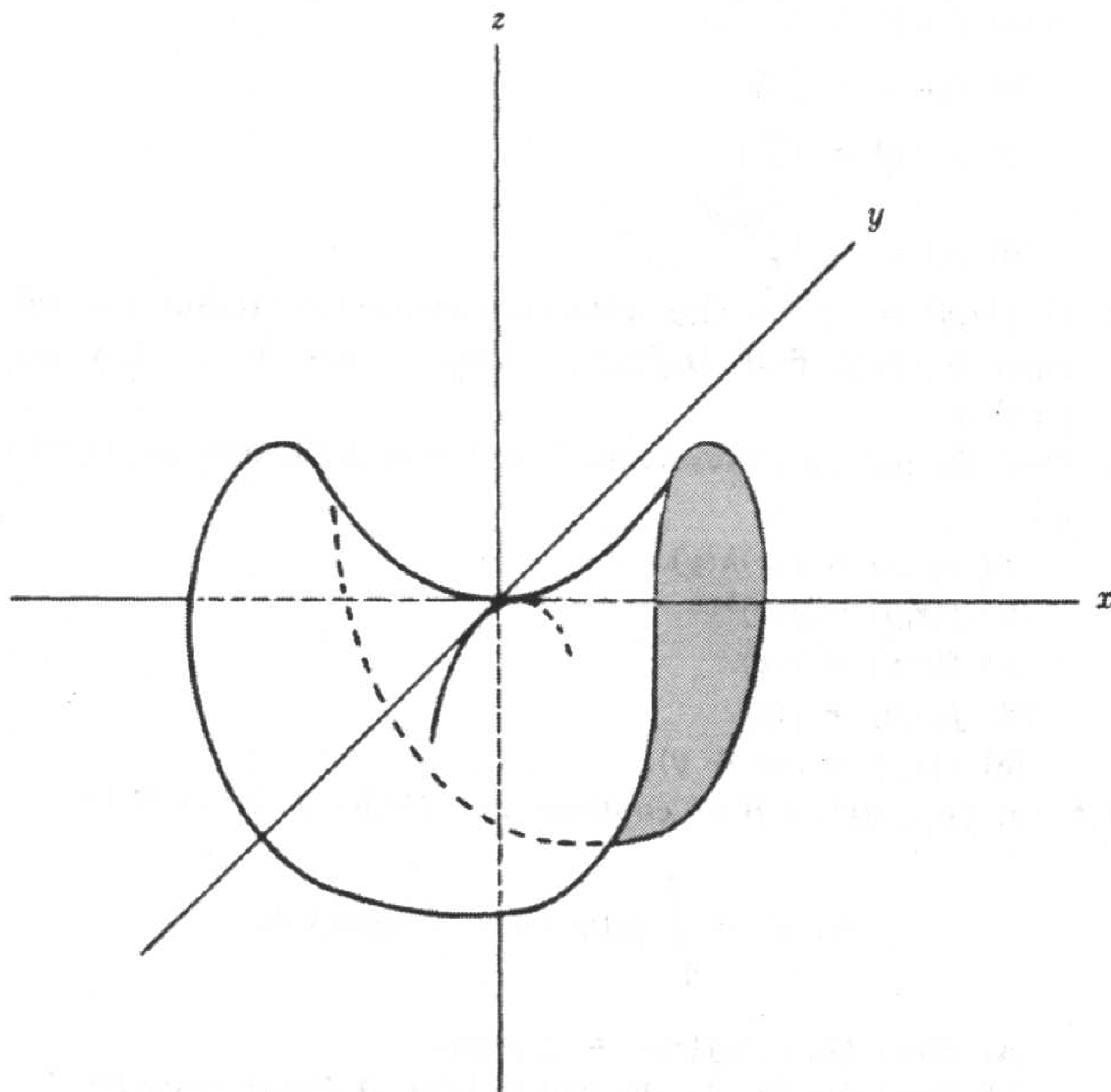


FIGURE 2-2

If Theorem 2-6 is used to find the maximum or minimum of f on A, the values of f at boundary points must be examined separately—a formidable task, since the boundary of A may be all of A! Problem 2-27 indicates one way of doing this, and Problem 5-16 states a superior method which can often be used.

Problems. **2-17.** Find the partial derivatives of the following functions:

(a) $f(x,y,z) = x^y$.
(b) $f(x,y,z) = z$.
(c) $f(x,y) = \sin(x \sin y)$.
(d) $f(x,y,z) = \sin(x \sin(y \sin z))$.
(e) $f(x,y,z) = x^{y^z}$.
(f) $f(x,y,z) = x^{y+z}$.
(g) $f(x,y,z) = (x + y)^z$.
(h) $f(x,y) = \sin(xy)$.
(i) $f(x,y) = [\sin(xy)]^{\cos 3}$.

2-18. Find the partial derivatives of the following functions (where $g\colon \mathbf{R} \to \mathbf{R}$ is continuous):

(a) $f(x,y) = \int_a^{x+y} g$.
(b) $f(x,y) = \int_y^x g$.
(c) $f(x,y) = \int_a^{xy} g$.
(d) $f(x,y) = \int_a^{\left(\int_b^y g\right)} g$.

2-19. If $f(x,y) = x^{x^{x^{x^y}}} + (\log x)(\arctan(\arctan(\arctan(\sin(\cos xy) - \log(x + y)))))$ find $D_2 f(1,y)$. *Hint:* There is an easy way to do this.

2-20. Find the partial derivatives of f in terms of the derivatives of g and h if

(a) $f(x,y) = g(x)h(y)$.
(b) $f(x,y) = g(x)^{h(y)}$.
(c) $f(x,y) = g(x)$.
(d) $f(x,y) = g(y)$.
(e) $f(x,y) = g(x + y)$.

2-21.* Let $g_1, g_2\colon \mathbf{R}^2 \to \mathbf{R}$ be continuous. Define $f\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \int_0^x g_1(t,0)\,dt + \int_0^y g_2(x,t)\,dt.$$

(a) Show that $D_2 f(x,y) = g_2(x,y)$.
(b) How should f be defined so that $D_1 f(x,y) = g_1(x,y)$?
(c) Find a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ such that $D_1 f(x,y) = x$ and $D_2 f(x,y) = y$. Find one such that $D_1 f(x,y) = y$ and $D_2 f(x,y) = x$.

2-22.* If $f: \mathbf{R}^2 \to \mathbf{R}$ and $D_2 f = 0$, show that f is independent of the second variable. If $D_1 f = D_2 f = 0$, show that f is constant.

2-23.* Let $A = \{(x,y) \in \mathbf{R}^2: x < 0, \text{ or } x \geq 0 \text{ and } y \neq 0\}$.

(a) If $f: A \to \mathbf{R}$ and $D_1 f = D_2 f = 0$, show that f is constant. *Hint:* Note that any two points in A can be connected by a sequence of lines each parallel to one of the axes.

(b) Find a function $f: A \to \mathbf{R}$ such that $D_2 f = 0$ but f is not independent of the second variable.

2-24. Define $f: \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

(a) Show that $D_2 f(x,0) = x$ for all x and $D_1 f(0,y) = -y$ for all y.

(b) Show that $D_{1,2} f(0,0) \neq D_{2,1} f(0,0)$.

2-25.* Define $f: \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is a C^∞ function, and $f^{(i)}(0) = 0$ for all i. *Hint:* The limit $f'(0) = \lim\limits_{h \to 0} \dfrac{e^{-h^{-2}}}{h} = \lim\limits_{h \to 0} \dfrac{1/h}{e^{h^{-2}}}$ can be evaluated by L'Hospital's rule. It is easy enough to find $f'(x)$ for $x \neq 0$, and $f''(0) = \lim\limits_{h \to 0} f'(h)/h$ can then be found by L'Hospital's rule.

2-26.* Let $$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1), \\ 0 & x \notin (-1,1). \end{cases}$$

(a) Show that $f: \mathbf{R} \to \mathbf{R}$ is a C^∞ function which is positive on $(-1,1)$ and 0 elsewhere.

(b) Show that there is a C^∞ function $g: \mathbf{R} \to [0,1]$ such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \varepsilon$. *Hint:* If f is a C^∞ function which is positive on $(0,\varepsilon)$ and 0 elsewhere, let $g(x) = \int_0^x f / \int_0^\varepsilon f$.

(c) If $a \in \mathbf{R}^n$, define $g: \mathbf{R}^n \to \mathbf{R}$ by

$$g(x) = f([x^1 - a^1]/\varepsilon) \cdot \ldots \cdot f([x^n - a^n]/\varepsilon).$$

Show that g is a C^∞ function which is positive on

$$(a^1 - \varepsilon, a^1 + \varepsilon) \times \cdots \times (a^n - \varepsilon, a^n + \varepsilon)$$

and zero elsewhere.

(d) If $A \subset \mathbf{R}^n$ is open and $C \subset A$ is compact, show that there is a non-negative C^∞ function $f: A \to \mathbf{R}$ such that $f(x) > 0$ for $x \in C$ and $f = 0$ outside of some closed set contained in A.

(e) Show that we can choose such an f so that $f: A \to [0,1]$ and $f(x) = 1$ for $x \in C$. *Hint:* If the function f of (d) satisfies $f(x) \geq \varepsilon$ for $x \in C$, consider $g \circ f$, where g is the function of (b).

2-27. Define $g, h: \{x \in \mathbf{R}^2: |x| \leq 1\} \to \mathbf{R}^3$ by

$$g(x,y) = (x,y, \sqrt{1 - x^2 - y^2}),$$
$$h(x,y) = (x,y, -\sqrt{1 - x^2 - y^2}).$$

Show that the maximum of f on $\{x \in \mathbf{R}^3: |x| = 1\}$ is either the maximum of $f \circ g$ or the maximum of $f \circ h$ on $\{x \in \mathbf{R}^2: |x| \leq 1\}$.

DERIVATIVES

The reader who has compared Problems 2-10 and 2-17 has probably already guessed the following.

2-7 ***Theorem.*** *If $f: \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a, then $D_jf^i(a)$ exists for $1 \leq i \leq m$, $1 \leq j \leq n$ and $f'(a)$ is the $m \times n$ matrix $(D_jf^i(a))$.*

Proof. Suppose first that $m = 1$, so that $f: \mathbf{R}^n \to \mathbf{R}$. Define $h: \mathbf{R} \to \mathbf{R}^n$ by $h(x) = (a^1, \ldots, x, \ldots, a^n)$, with x in the jth place. Then $D_jf(a) = (f \circ h)'(a^j)$. Hence, by Theorem 2-2,

$$(f \circ h)'(a^j) = f'(a) \cdot h'(a^j)$$
$$= f'(a) \cdot \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \leftarrow j\text{th place.}$$

Since $(f \circ h)'(a^j)$ has the single entry $D_jf(a)$, this shows that $D_jf(a)$ exists and is the jth entry of the $1 \times n$ matrix $f'(a)$.

The theorem now follows for arbitrary m since, by Theorem 2-3, each f^i is differentiable and the ith row of $f'(a)$ is $(f^i)'(a)$. ∎

There are several examples in the problems to show that the converse of Theorem 2-7 is false. It is true, however, if one hypothesis is added.

2-8 Theorem. *If $f\colon \mathbf{R}^n \to \mathbf{R}^m$, then $Df(a)$ exists if all $D_jf^i(x)$ exist in an open set containing a and if each function D_jf^i is continuous at a.*
(Such a function f is called **continuously differentiable** at a.)

Proof. As in the proof of Theorem 2-7, it suffices to consider the case $m = 1$, so that $f\colon \mathbf{R}^n \to \mathbf{R}$. Then

$$\begin{aligned} f(a+h) - f(a) = {} & f(a^1 + h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n) \\ & + f(a^1 + h^1, a^2 + h^2, a^3, \ldots, a^n) \\ & \qquad - f(a^1 + h^1, a^2, \ldots, a^n) \\ & + \cdots \\ & + f(a^1 + h^1, \ldots, a^n + h^n) \\ & \qquad - f(a^1 + h^1, \ldots, a^{n-1} + h^{n-1}, a^n). \end{aligned}$$

Recall that D_1f is the derivative of the function g defined by $g(x) = f(x, a^2, \ldots, a^n)$. Applying the mean-value theorem to g we obtain

$$\begin{aligned} f(a^1 + h^1, a^2, \ldots, a^n) & - f(a^1, \ldots, a^n) \\ & = h^1 \cdot D_1f(b_1, a^2, \ldots, a^n) \end{aligned}$$

for some b_1 between a^1 and $a^1 + h^1$. Similarly the ith term in the sum equals

$$h^i \cdot D_if(a^1 + h^1, \ldots, a^{i-1} + h^{i-1}, b_i, \ldots, a^n) = h^iD_if(c_i),$$

for some c_i. Then

$$\begin{aligned} \lim_{h \to 0} \frac{\left| f(a+h) - f(a) - \sum_{i=1}^n D_if(a) \cdot h^i \right|}{|h|} & = \lim_{h \to 0} \frac{\left| \sum_{i=1}^n [D_if(c_i) - D_if(a)] \cdot h^i \right|}{|h|} \\ & \le \lim_{h \to 0} \sum_{i=1}^n |D_if(c_i) - D_if(a)| \cdot \frac{|h^i|}{|h|} \\ & \le \lim_{h \to 0} \sum_{i=1}^n |D_if(c_i) - D_if(a)| \\ & = 0, \end{aligned}$$

since D_if is continuous at a. ∎

Although the chain rule was used in the proof of Theorem 2-7, it could easily have been eliminated. With Theorem 2-8 to provide differentiable functions, and Theorem 2-7 to provide their derivatives, the chain rule may therefore seem almost superfluous. However, it has an extremely important corollary concerning partial derivatives.

2-9 Theorem. *Let* $g_1, \ldots, g_m\colon \mathbf{R}^n \to \mathbf{R}$ *be continuously differentiable at* a, *and let* $f\colon \mathbf{R}^m \to \mathbf{R}$ *be differentiable at* $(g_1(a), \ldots, g_m(a))$. *Define the function* $F\colon \mathbf{R}^n \to \mathbf{R}$ *by* $F(x) = f(g_1(x), \ldots, g_m(x))$. *Then*

$$D_iF(a) = \sum_{j=1}^{m} D_jf(g_1(a), \ldots, g_m(a)) \cdot D_ig_j(a).$$

Proof. The function F is just the composition $f \circ g$, where $g = (g_1, \ldots, g_m)$. Since g_i is continuously differentiable at a, it follows from Theorem 2-8 that g is differentiable at a. Hence by Theorem 2-2,

$$F'(a) = f'(g(a)) \cdot g'(a) =$$

$$(D_1f(g(a)), \ldots, D_mf(g(a))) \cdot \begin{pmatrix} D_1g_1(a), & \cdots & ,D_ng_1(a) \\ \vdots & & \vdots \\ D_1g_m(a), & \ldots & ,D_ng_m(a) \end{pmatrix}$$

But $D_iF(a)$ is the ith entry of the left side of this equation, while $\sum_{j=1}^m D_jf(g_1(a), \ldots, g_m(a)) \cdot D_ig_j(a)$ is the ith entry of the right side. ▌

Theorem 2-9 is often called the *chain rule*, but is weaker than Theorem 2-2 since g could be differentiable without g_i being continuously differentiable (see Problem 2-32). Most computations requiring Theorem 2-9 are fairly straightforward. A slight subtlety is required for the function $F\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$F(x,y) = f(g(x,y),h(x),k(y))$$

where $h,k\colon \mathbf{R} \to \mathbf{R}$. In order to apply Theorem 2-9 define $\bar{h},\bar{k}\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$\bar{h}(x,y) = h(x) \qquad \bar{k}(x,y) = k(y).$$

Then

$$\begin{aligned} D_1\bar{h}(x,y) &= h'(x) & D_2\bar{h}(x,y) &= 0, \\ D_1\bar{k}(x,y) &= 0 & D_2\bar{k}(x,y) &= k'(y), \end{aligned}$$

and we can write

$$F(x,y) = f(g(x,y),\bar{h}(x,y),\bar{k}(x,y)).$$

Letting $a = (g(x,y),h(x),k(y))$, we obtain

$$\begin{aligned} D_1F(x,y) &= D_1f(a) \cdot D_1g(x,y) + D_2f(a) \cdot h'(x), \\ D_2F(x,y) &= D_1f(a) \cdot D_2g(x,y) + D_3f(a) \cdot k'(y). \end{aligned}$$

It should, of course, be unnecessary for you to actually write down the functions $\bar{h}$ and $\bar{k}$.

Problems. **2-28.** Find expressions for the partial derivatives of the following functions:

(a) $F(x,y) = f(g(x)k(y),\ g(x) + h(y))$.
(b) $F(x,y,z) = f(g(x + y),\ h(y + z))$.
(c) $F(x,y,z) = f(x^y,y^z,z^x)$.
(d) $F(x,y) = f(x,g(x),h(x,y))$.

2-29. Let $f\colon \mathbf{R}^n \to \mathbf{R}$. For $x \in \mathbf{R}^n$, the limit

$$\lim_{t\to 0} \frac{f(a + tx) - f(a)}{t},$$

if it exists, is denoted $D_xf(a)$, and called the **directional derivative** of f at a, in the direction x.

(a) Show that $D_{e_i}f(a) = D_if(a)$.
(b) Show that $D_{tx}f(a) = tD_xf(a)$.
(c) If f is differentiable at a, show that $D_xf(a) = Df(a)(x)$ and therefore $D_{x+y}f(a) = D_xf(a) + D_yf(a)$.

2-30. Let f be defined as in Problem 2-4. Show that $D_xf(0,0)$ exists for all x, but if $g \neq 0$, then $D_{x+y}f(0,0) = D_xf(0,0) + D_yf(0,0)$ is not true for all x and y.

2-31. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be defined as in Problem 1-26. Show that $D_xf(0,0)$ exists for all x, although f is not even continuous at $(0,0)$.

2-32. (a) Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at 0 but f' is not continuous at 0.

(b) Let $f: \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{\sqrt{x^2 + y^2}} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

Show that f is differentiable at (0,0) but $D_i f$ is not continuous at (0,0).

2-33. Show that the continuity of $D_1 f^j$ at a may be eliminated from the hypothesis of Theorem 2-8.

2-34. A function $f: \mathbf{R}^n \to \mathbf{R}$ is **homogeneous** of degree m if $f(tx) = t^m f(x)$ for all x. If f is also differentiable, show that

$$\sum_{i=1}^{n} x^i D_i f(x) = m f(x).$$

Hint: If $g(t) = f(tx)$, find $g'(1)$.

2-35. If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable and $f(0) = 0$, prove that there exist $g_i: \mathbf{R}^n \to \mathbf{R}$ such that

$$f(x) = \sum_{i=1}^{n} x^i g_i(x).$$

Hint: If $h_x(t) = f(tx)$, then $f(x) = \int_0^1 h_x'(t)dt$.

INVERSE FUNCTIONS

Suppose that $f: \mathbf{R} \to \mathbf{R}$ is continuously differentiable in an open set containing a and $f'(a) \neq 0$. If $f'(a) > 0$, there is an open interval V containing a such that $f'(x) > 0$ for $x \in V$, and a similar statement holds if $f'(a) < 0$. Thus f is increasing (or decreasing) on V, and is therefore 1-1 with an inverse function f^{-1} defined on some open interval W containing $f(a)$. Moreover it is not hard to show that f^{-1} is differentiable, and for $y \in W$ that

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

An analogous discussion in higher dimensions is much more involved, but the result (Theorem 2-11) is very important. We begin with a simple lemma.

2-10 Lemma. *Let $A \subset \mathbf{R}^n$ be a rectangle and let $f\colon A \to \mathbf{R}^n$ be continuously differentiable. If there is a number M such that $|D_j f^i(x)| \le M$ for all x in the interior of A, then*

$$|f(x) - f(y)| \le n^2 M |x - y|$$

for all $x,y \in A$.

Proof. We have

$$f^i(y) - f^i(x) = \sum_{j=1}^{n} [f^i(y^1, \ldots, y^j, x^{j+1}, \ldots, x^n) - f^i(y^1, \ldots, y^{j-1}, x^j, \ldots, x^n)].$$

Applying the mean-value theorem we obtain

$$f^i(y^1, \ldots, y^j, x^{j+1}, \ldots, x^n) - f^i(y^1, \ldots, y^{j-1}, x^j, \ldots, x^n) = (y^j - x^j) \cdot D_j f^i(z_{ij})$$

for some z_{ij}. The expression on the right has absolute value less than or equal to $M \cdot |y^j - x^j|$. Thus

$$|f^i(y) - f^i(x)| \le \sum_{j=1}^{n} |y^j - x^j| \cdot M \le nM|y - x|$$

since each $|y^j - x^j| \le |y - x|$. Finally

$$|f(y) - f(x)| \le \sum_{i=1}^{n} |f^i(y) - f^i(x)| \le n^2 M \cdot |y - x|. \quad \blacksquare$$

2-11 Theorem (Inverse Function Theorem). *Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is continuously differentiable in an open set containing a, and $\det f'(a) \ne 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f\colon V \to W$ has a continuous inverse $f^{-1}\colon W \to V$ which is differentiable and for all $y \in W$ satisfies*

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}.$$

Proof. Let λ be the linear transformation $Df(a)$. Then λ is non-singular, since $\det f'(a) \ne 0$. Now $D(\lambda^{-1} \circ f)(a) = D(\lambda^{-1})(f(a)) \circ Df(a) = \lambda^{-1} \circ Df(a)$ is the identity linear

transformation. If the theorem is true for $\lambda^{-1} \circ f$, it is clearly true for f. Therefore we may assume at the outset that λ is the identity. Thus whenever $f(a + h) = f(a)$, we have

$$\frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = \frac{|h|}{|h|} = 1.$$

But

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

This means that we cannot have $f(x) = f(a)$ for x arbitrarily close to, but unequal to, a. Therefore there is a closed rectangle U containing a in its interior such that

1. $f(x) \neq f(a)$ if $x \in U$ and $x \neq a$.

Since f is continuously differentiable in an open set containing a, we can also assume that

2. $\det f'(x) \neq 0$ for $x \in U$.
3. $|D_jf^i(x) - D_jf^i(a)| < 1/2n^2$ for all i, j, and $x \in U$.

Note that (3) and Lemma 2-10 applied to $g(x) = f(x) - x$ imply for $x_1, x_2 \in U$ that

$$|f(x_1) - x_1 - (f(x_2) - x_2)| \leq \tfrac{1}{2}|x_1 - x_2|.$$

Since

$$\begin{aligned} |x_1 - x_2| - |f(x_1) - f(x_2)| &\leq |f(x_1) - x_1 - (f(x_2) - x_2)| \\ &\leq \tfrac{1}{2}|x_1 - x_2|, \end{aligned}$$

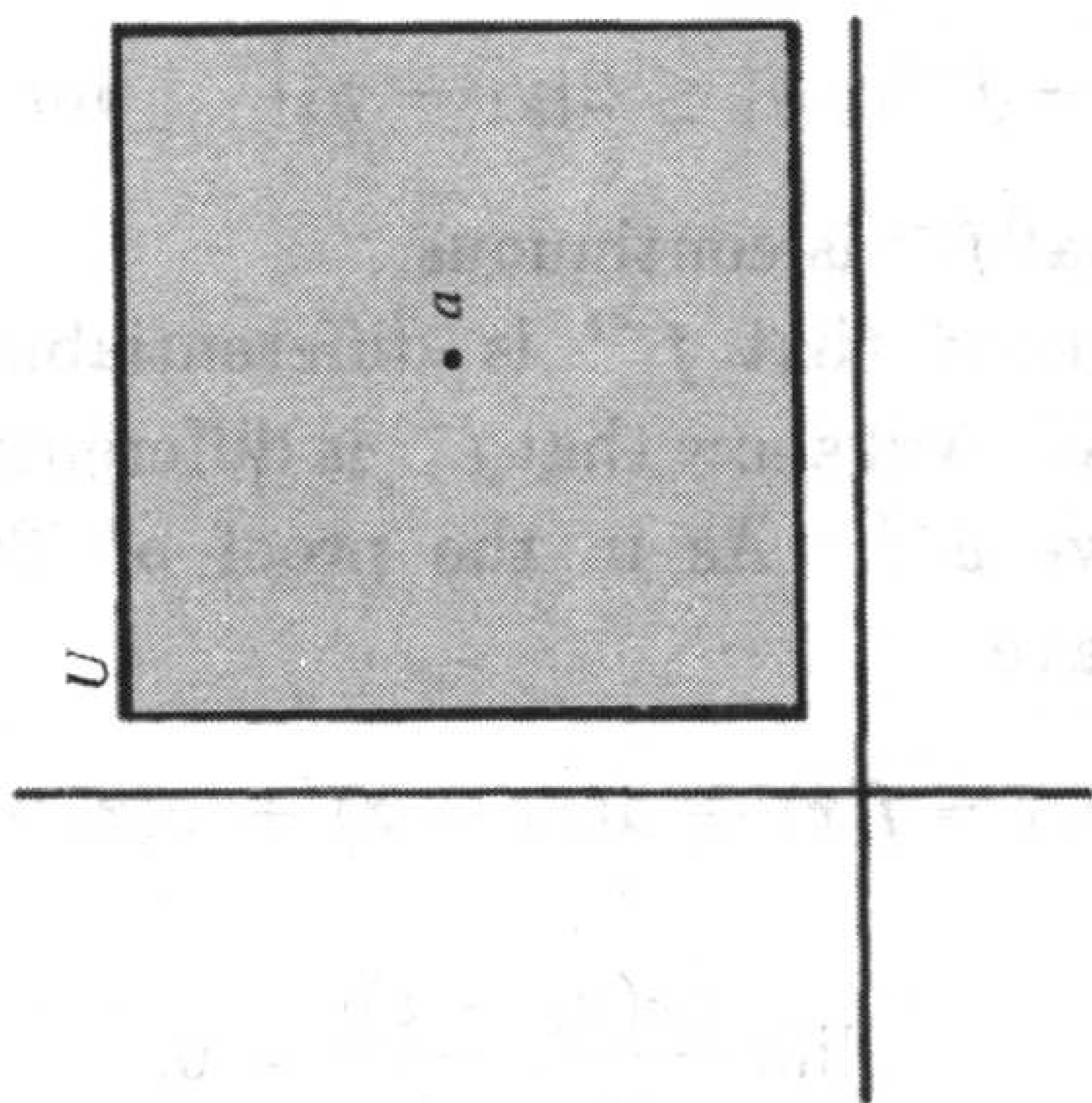
we obtain

4. $|x_1 - x_2| \leq 2|f(x_1) - f(x_2)|$ for $x_1, x_2 \in U$.

Now f(boundary U) is a compact set which, by (1), does not contain $f(a)$ (Figure 2-3). Therefore there is a number $d > 0$ such that $|f(a) - f(x)| \geq d$ for $x \in$ boundary U. Let $W = \{y: |y - f(a)| < d/2\}$. If $y \in W$ and $x \in$ boundary U, then

5. $|y - f(a)| < |y - f(x)|$.

We will show that for any $y \in W$ there is a unique x in interior U such that $f(x) = y$. To prove this consider the



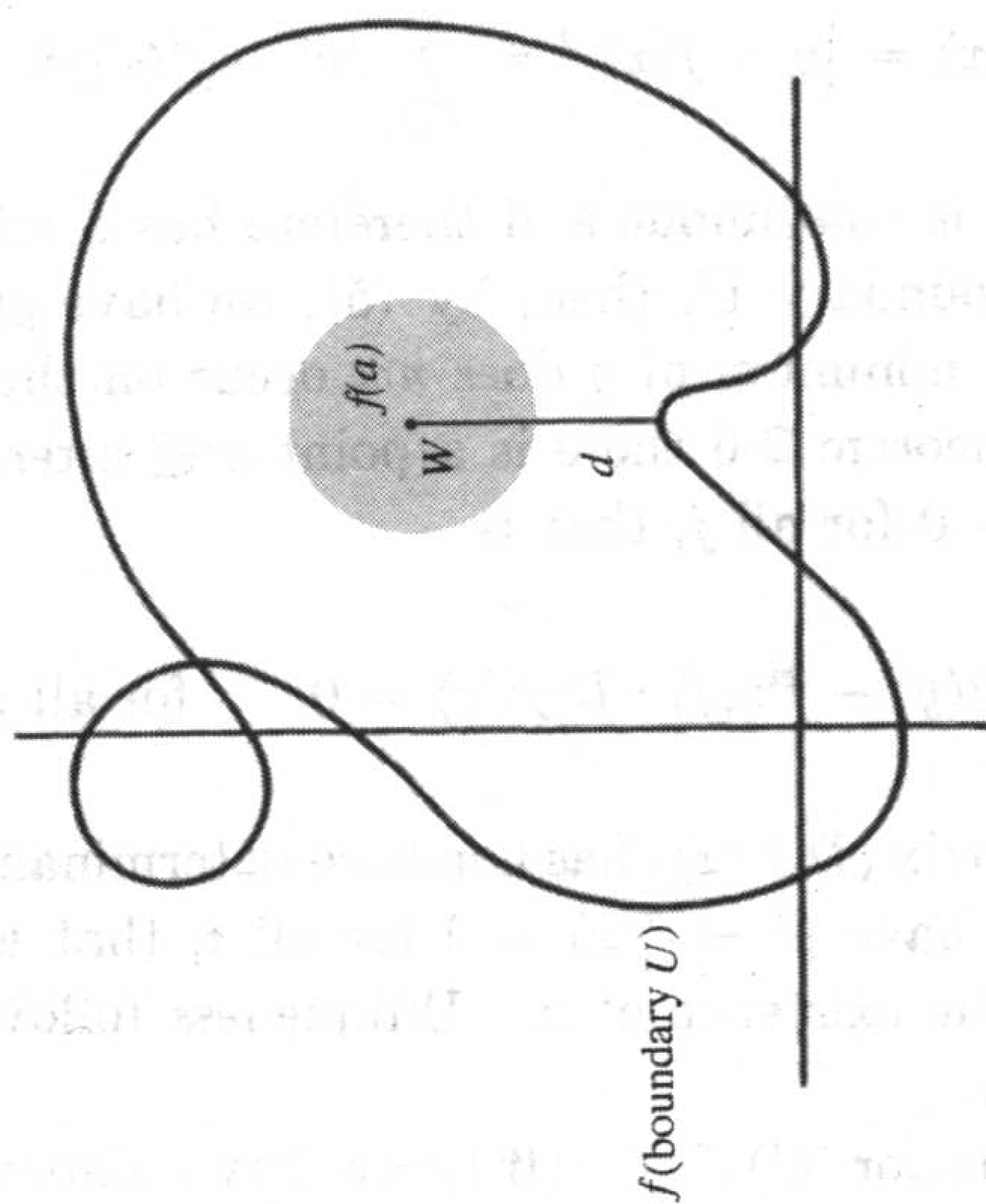


FIGURE 2-3

function $g\colon U \to \mathbf{R}$ defined by

$$g(x) = |y - f(x)|^2 = \sum_{i=1}^{n} (y^i - f^i(x))^2.$$

This function is continuous and therefore has a minimum on U. If $x \in$ boundary U, then, by (5), we have $g(a) < g(x)$. Therefore the minimum of g does *not* occur on the boundary of U. By Theorem 2-6 there is a point $x \in$ interior U such that $D_jg(x) = 0$ for all j, that is

$$\sum_{i=1}^{n} 2(y^i - f^i(x)) \cdot D_jf^i(x) = 0 \qquad \text{for all } j.$$

By (2) the matrix $(D_jf^i(x))$ has non-zero determinant. Therefore we must have $y^i - f^i(x) = 0$ for all i, that is $y = f(x)$. This proves the existence of x. Uniqueness follows immediately from (4).

If $V = (\text{interior } U) \cap f^{-1}(W)$, we have shown that the function $f\colon V \to W$ has an inverse $f^{-1}\colon W \to V$. We can rewrite (4) as

6. $|f^{-1}(y_1) - f^{-1}(y_2)| \le 2|y_1 - y_2| \qquad \text{for } y_1, y_2 \in W.$

This shows that f^{-1} is continuous.

Only the proof that f^{-1} is differentiable remains. Let $\mu = Df(x)$. We will show that f^{-1} is differentiable at $y = f(x)$ with derivative μ^{-1}. As in the proof of Theorem 2-2, for $x_1 \in V$, we have

$$f(x_1) = f(x) + \mu(x_1 - x) + \varphi(x_1 - x),$$

where

$$\lim_{x_1 \to x} \frac{|\varphi(x_1 - x)|}{|x_1 - x|} = 0.$$

Therefore

$$\mu^{-1}(f(x_1) - f(x)) = x_1 - x + \mu^{-1}(\varphi(x_1 - x)).$$

Since every $y_1 \in W$ is of the form $f(x_1)$ for some $x_1 \in V$, this can be written

$$f^{-1}(y_1) = f^{-1}(y) + \mu^{-1}(y_1 - y) - \mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y))),$$

and it therefore suffices to show that

$$\lim_{y_1 \to y} \frac{|\mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y)))|}{|y_1 - y|} = 0.$$

Therefore (Problem 1-10) it suffices to show that

$$\lim_{y_1 \to y} \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|} = 0.$$

Now

$$\frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|} = \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|} \cdot \frac{|f^{-1}(y_1) - f^{-1}(y)|}{|y_1 - y|}.$$

Since f^{-1} is continuous, $f^{-1}(y_1) \to f^{-1}(y)$ as $y_1 \to y$. Therefore the first factor approaches 0. Since, by (6), the second factor is less than 2, the product also approaches 0. ▌

It should be noted that an inverse function f^{-1} may exist even if $\det f'(a) = 0$. For example, if $f\colon \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^3$, then $f'(0) = 0$ but f has the inverse function $f^{-1}(x) = \sqrt[3]{x}$. One thing is certain however: if $\det f'(a) = 0$, then f^{-1} cannot be differentiable at $f(a)$. To prove this note that $f \circ f^{-1}(x) = x$. If f^{-1} were differentiable at $f(a)$, the chain rule would give $f'(a) \cdot (f^{-1})'(f(a)) = I$, and consequently $\det f'(a) \cdot \det(f^{-1})'(f(a)) = 1$, contradicting $\det f'(a) = 0$.

Problems. **2-36.*** Let $A \subset \mathbf{R}^n$ be an open set and $f\colon A \to \mathbf{R}^n$ a continuously differentiable 1-1 function such that $\det f'(x) \neq 0$ for all x. Show that $f(A)$ is an open set and $f^{-1}\colon f(A) \to A$ is differentiable. Show also that $f(B)$ is open for any open set $B \subset A$.

2-37. (a) Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be a continuously differentiable function. Show that f is *not* 1-1. *Hint:* If, for example, $D_1f(x,y) \neq 0$ for all (x,y) in some open set A, consider $g\colon A \to \mathbf{R}^2$ defined by $g(x,y) = (f(x,y),y)$.

(b) Generalize this result to the case of a continuously differentiable function $f\colon \mathbf{R}^n \to \mathbf{R}^m$ with $m < n$.

2-38. (a) If $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbf{R}$, show that f is 1-1 (on all of $\mathbf{R}$).

(b) Define $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) = (e^x \cos y, e^x \sin y)$. Show that $\det f'(x,y) \neq 0$ for all (x,y) but f is not 1-1.

2-39. Use the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x}{2} + x^2 \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

to show that continuity of the derivative cannot be eliminated from the hypothesis of Theorem 2-11.

IMPLICIT FUNCTIONS

Consider the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x,y) = x^2 + y^2 - 1$. If we choose (a,b) with $f(a,b) = 0$ and $a \neq 1, -1$, there are (Figure 2-4) open intervals A containing a and B containing b with the following property: if $x \in A$, there is a unique $y \in B$ with $f(x,y) = 0$. We can therefore define

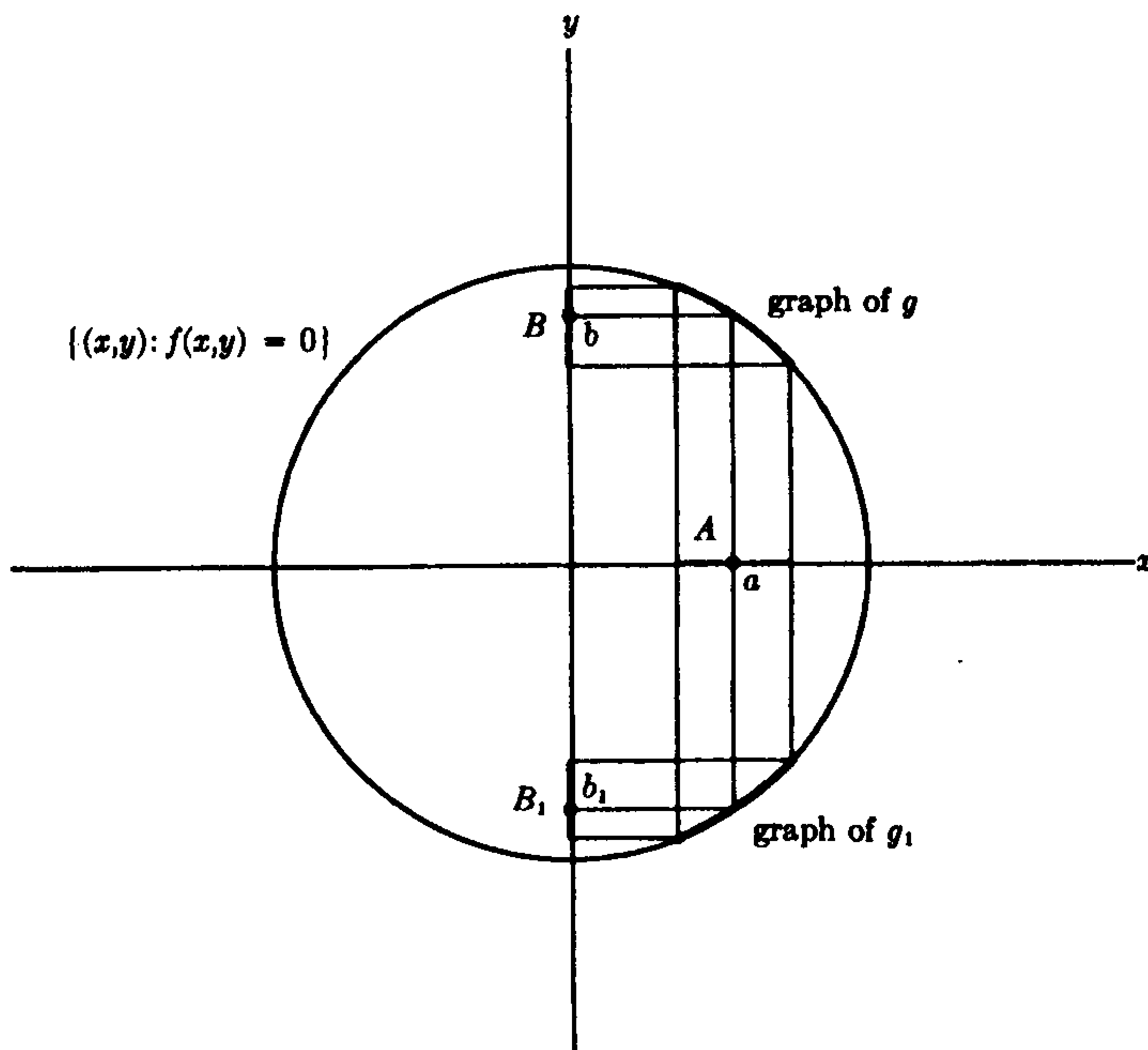


FIGURE 2-4

a function $g\colon A \to \mathbf{R}$ by the condition $g(x) \in B$ and $f(x,g(x)) = 0$ (if $b > 0$, as indicated in Figure 2-4, then $g(x) = \sqrt{1 - x^2}$). For the function f we are considering there is another number b_1 such that $f(a,b_1) = 0$. There will also be an interval B_1 containing b_1 such that, when $x \in A$, we have $f(x,g_1(x)) = 0$ for a unique $g_1(x) \in B_1$ (here $g_1(x) = -\sqrt{1 - x^2}$). Both g and g_1 are differentiable. These functions are said to be defined **implicitly** by the equation $f(x,y) = 0$.

If we choose $a = 1$ or -1 it is impossible to find any such function g defined in an open interval containing a. We would like a simple criterion for deciding when, in general, such a function can be found. More generally we may ask the following: If $f\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ and $f(a^1, \ldots, a^n, b) = 0$, when can we find, for each $(x^1, \ldots, x^n)$ near $(a^1, \ldots, a^n)$, a unique y near b such that $f(x^1, \ldots, x^n, y) = 0$? Even more generally, we can ask about the possibility of solving m equations, depending upon parameters $x^1, \ldots, x^n$, in m unknowns: If

$$f_i\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R} \qquad i = 1, \ldots, m$$

and

$$f_i(a^1, \ldots, a^n, b^1, \ldots, b^m) = 0 \qquad i = 1, \ldots, m,$$

when can we find, for each $(x^1, \ldots, x^n)$ near $(a^1, \ldots, a^n)$ a unique $(y^1, \ldots, y^m)$ near $(b^1, \ldots, b^m)$ which satisfies $f_i(x^1, \ldots, x^n, y^1, \ldots, y^m) = 0$? The answer is provided by

2-12 Theorem (Implicit Function Theorem). *Suppose $f\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$ is continuously differentiable in an open set containing (a,b) and $f(a,b) = 0$. Let M be the $m \times m$ matrix*

$$(D_{n+j}f^i(a,b)) \qquad 1 \le i, j \le m.$$

If $\det M \neq 0$, there is an open set $A \subset \mathbf{R}^n$ containing a and an open set $B \subset \mathbf{R}^m$ containing b, with the following property: for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x,g(x)) = 0$. The function g is differentiable.

Proof. Define $F\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^m$ by $F(x,y) = (x,f(x,y))$. Then $\det F'(a,b) = \det M \neq 0$. By Theorem 2-11 there is an open set $W \subset \mathbf{R}^n \times \mathbf{R}^m$ containing $F(a,b) = (a,0)$ and an open set in $\mathbf{R}^n \times \mathbf{R}^m$ containing (a,b), which we may take to be of the form $A \times B$, such that $F\colon A \times B \to W$ has a differentiable inverse $h\colon W \to A \times B$. Clearly h is of the form $h(x,y) = (x,k(x,y))$ for some differentiable function k (since F is of this form). Let $\pi\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$ be defined by $\pi(x,y) = y$; then $\pi \circ F = f$. Therefore

$$\begin{aligned} f(x,k(x,y)) &= f \circ h(x,y) = (\pi \circ F) \circ h(x,y) \\ &= \pi \circ (F \circ h)(x,y) = \pi(x,y) = y. \end{aligned}$$

Thus $f(x,k(x,0)) = 0$; in other words we can define $g(x) = k(x,0)$. ▮

Since the function g is known to be differentiable, it is easy to find its derivative. In fact, since $f^i(x,g(x)) = 0$, taking D_j of both sides gives

$$0 = D_j f^i(x,g(x)) + \sum_{\alpha=1}^{m} D_{n+\alpha} f^i(x,g(x)) \cdot D_j g^\alpha(x) \qquad i,j = 1, \ldots, m.$$

Since $\det M \neq 0$, these equations can be solved for $D_j g^\alpha(x)$. The answer will depend on the various $D_j f^i(x,g(x))$, and therefore on $g(x)$. This is unavoidable, since the function g is not unique. Reconsidering the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x,y) = x^2 + y^2 - 1$, we note that two possible functions satisfying $f(x,g(x)) = 0$ are $g(x) = \sqrt{1 - x^2}$ and $g(x) = -\sqrt{1 - x^2}$. Differentiating $f(x,g(x)) = 0$ gives

$$D_1 f(x,g(x)) + D_2 f(x,g(x)) \cdot g'(x) = 0,$$

or

$$\begin{aligned} 2x + 2g(x) \cdot g'(x) &= 0, \\ g'(x) &= -x/g(x), \end{aligned}$$

which is indeed the case for either $g(x) = \sqrt{1 - x^2}$ or $g(x) = -\sqrt{1 - x^2}$.

A generalization of the argument for Theorem 2-12 can be given, which will be important in Chapter 5.

2-13 Theorem. *Let $f: \mathbf{R}^n \to \mathbf{R}^p$ be continuously differentiable in an open set containing a, where $p \le n$. If $f(a) = 0$ and the $p \times n$ matrix $(D_j f^i(a))$ has rank p, then there is an open set $A \subset \mathbf{R}^n$ containing a and a differentiable function $h: A \to \mathbf{R}^n$ with differentiable inverse such that*

$$f \circ h(x^1, \ldots, x^n) = (x^{n-p+1}, \ldots, x^n).$$

Proof. We can consider f as a function $f: \mathbf{R}^{n-p} \times \mathbf{R}^p \to \mathbf{R}^p$. If det $M \neq 0$, then M is the $p \times p$ matrix $(D_{n-p+j} f^i(a))$, $1 \le i, j \le p$, then we are precisely in the situation considered in the proof of Theorem 2-12, and as we showed in that proof, there is h such that $f \circ h(x^1, \ldots, x^n) = (x^{n-p+1}, \ldots, x^n)$.

In general, since $(D_j f^i(a))$ has rank p, there will be $j_1 < \cdots < j_p$ such that the matrix $(D_j f^i(a))$ $1 \le i \le p$, $j = j_1, \ldots, j_p$ has non-zero determinant. If $g: \mathbf{R}^n \to \mathbf{R}^n$ permutes the x^j so that $g(x^1, \ldots, x^n) = (\ldots, x^{j_1}, \ldots, x^{j_p})$, then $f \circ g$ is a function of the type already considered, so $((f \circ g) \circ k)(x^1, \ldots, x^n) = (x^{n-p+1}, \ldots, x^n)$ for some k. Let $h = g \circ k$. ▌

Problems. 2-40. Use the implicit function theorem to re-do Problem 2-15(c).

2-41. Let $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be differentiable. For each $x \in \mathbf{R}$ define $g_x: \mathbf{R} \to \mathbf{R}$ by $g_x(y) = f(x,y)$. Suppose that for each x there is a unique y with $g_x'(y) = 0$; let $c(x)$ be this y.

(a) If $D_{2,2}f(x,y) \neq 0$ for all (x,y), show that c is differentiable and

$$c'(x) = -\frac{D_{2,1}f(x,c(x))}{D_{2,2}f(x,c(x))}.$$

Hint: $g_x'(y) = 0$ can be written $D_2 f(x,y) = 0$.

(b) Show that if $c'(x) = 0$, then for some y we have

$$D_{2,1}f(x,y) = 0,$$
$$D_2 f(x,y) = 0.$$

(c) Let $f(x,y) = x(y \log y - y) - y \log x$. Find

$$\max_{\frac{1}{2} \le x \le 2} \left(\min_{\frac{1}{3} \le y \le 1} f(x,y) \right).$$

NOTATION

This section is a brief and not entirely unprejudiced discussion of classical notation connected with partial derivatives.

The partial derivative $D_1f(x,y,z)$ is denoted, among devotees of classical notation, by

$$\frac{\partial f(x,y,z)}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x}(x,y,z) \quad \text{or} \quad \frac{\partial}{\partial x}f(x,y,z)$$

or any other convenient similar symbol. This notation forces one to write

$$\frac{\partial f}{\partial u}(u,v,w)$$

for $D_1f(u,v,w)$, although the symbol

$$\left.\frac{\partial f(x,y,z)}{\partial x}\right|_{(x,y,z)=(u,v,w)} \quad \text{or} \quad \frac{\partial f(x,y,z)}{\partial x}(u,v,w)$$

or something similar may be used (and must be used for an expression like $D_1f(7,3,2)$). Similar notation is used for D_2f and D_3f. Higher-order derivatives are denoted by symbols like

$$D_2D_1f(x,y,z) = \frac{\partial^2 f(x,y,z)}{\partial y\, \partial x}.$$

When $f\colon \mathbf{R} \to \mathbf{R}$, the symbol ∂ automatically reverts to d; thus

$$\frac{d \sin x}{dx}, \quad \text{not} \quad \frac{\partial \sin x}{\partial x}.$$

The mere statement of Theorem 2-2 in classical notation requires the introduction of irrelevant letters. The usual evaluation for $D_1(f \circ (g,h))$ runs as follows:

If $f(u,v)$ is a function and $u = g(x,y)$ and $v = h(x,y)$, then

$$\frac{\partial f(g(x,y),\, h(x,y))}{\partial x} = \frac{\partial f(u,v)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f(u,v)}{\partial v}\frac{\partial v}{\partial x}.$$

[The symbol $\partial u/\partial x$ means $\partial/\partial x\, g(x,y)$ and $\partial/\partial u\, f(u,v)$ means

$D_1f(u,v) = D_1f(g(x,y), h(x,y))$.] This equation is often written simply

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

Note that f means something different on the two sides of the equation!

The notation df/dx, always a little too tempting, has inspired many (usually meaningless) definitions of dx and df separately, the sole purpose of which is to make the equation

$$df = \frac{df}{dx} \cdot dx$$

work out. If $f\colon \mathbf{R}^2 \to \mathbf{R}$ then df is *defined*, classically, as

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

(whatever dx and dy mean).

Chapter 4 contains rigorous definitions which enable us to prove the above equations as theorems. It is a touchy question whether or not these modern definitions represent a real improvement over classical formalism; this the reader must decide for himself.

3

Integration

BASIC DEFINITIONS

The definition of the integral of a function $f: A \to \mathbf{R}$, where $A \subset \mathbf{R}^n$ is a closed rectangle, is so similar to that of the ordinary integral that a rapid treatment will be given.

Recall that a partition P of a closed interval $[a,b]$ is a sequence $t_0, \ldots, t_k$, where $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$. The partition P divides the interval $[a,b]$ into k subintervals $[t_{i-1},t_i]$. A **partition** of a rectangle $[a_1,b_1] \times \cdots \times [a_n,b_n]$ is a collection $P = (P_1, \ldots, P_n)$, where each P_i is a partition of the interval $[a_i,b_i]$. Suppose, for example, that $P_1 = t_0, \ldots, t_k$ is a partition of $[a_1,b_1]$ and $P_2 = s_0, \ldots, s_l$ is a partition of $[a_2,b_2]$. Then the partition $P = (P_1,P_2)$ of $[a_1,b_1] \times [a_2,b_2]$ divides the closed rectangle $[a_1,b_1] \times [a_2,b_2]$ into $k \cdot l$ subrectangles, a typical one being $[t_{i-1},t_i] \times [s_{j-1},s_j]$. In general, if P_i divides $[a_i,b_i]$ into N_i subintervals, then $P = (P_1, \ldots, P_n)$ divides $[a_1,b_1] \times \cdots \times [a_n,b_n]$ into $N = N_1 \cdot \ldots \cdot N_n$ subrectangles. These subrectangles will be called **subrectangles of the partition** P.

Suppose now that A is a rectangle, $f: A \to \mathbf{R}$ is a bounded

function, and P is a partition of A. For each subrectangle S of the partition let

$$m_S(f) = \inf\{f(x): x \in S\},$$
$$M_S(f) = \sup\{f(x): x \in S\},$$

and let $v(S)$ be the volume of S [the **volume** of a rectangle $[a_1,b_1] \times \cdots \times [a_n,b_n]$, and also of $(a_1,b_1) \times \cdots \times (a_n,b_n)$, is defined as $(b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$]. The **lower** and **upper sums** of f for P are defined by

$$L(f,P) = \sum_S m_S(f) \cdot v(S) \quad \text{and} \quad U(f,P) = \sum_S M_S(f) \cdot v(S).$$

Clearly $L(f,P) \le U(f,P)$, and an even stronger assertion (3-2) is true.

3-1 Lemma. *Suppose the partition P' refines P (that is, each subrectangle of P' is contained in a subrectangle of P). Then*

$$L(f,P) \le L(f,P') \quad \textit{and} \quad U(f,P') \le U(f,P).$$

Proof. Each subrectangle S of P is divided into several subrectangles $S_1, \ldots, S_\alpha$ of P', so $v(S) = v(S_1) + \cdots + v(S_\alpha)$. Now $m_S(f) \le m_{S_i}(f)$, since the values $f(x)$ for $x \in S$ include all values $f(x)$ for $x \in S_i$ (and possibly smaller ones). Thus

$$\begin{aligned} m_S(f) \cdot v(S) &= m_S(f) \cdot v(S_1) + \cdots + m_S(f) \cdot v(S_\alpha) \\ &\le m_{S_1}(f) \cdot v(S_1) + \cdots + m_{S_\alpha}(f) \cdot v(S_\alpha). \end{aligned}$$

The sum, for all S, of the terms on the left side is $L(f,P)$, while the sum of all the terms on the right side is $L(f,P')$. Hence $L(f,P) \le L(f,P')$. The proof for upper sums is similar. ▮

3-2 Corollary. *If P and P' are any two partitions, then $L(f,P') \le U(f,P)$.*

Proof. Let P'' be a partition which refines both P and P'. (For example, let $P'' = (P''_1, \ldots, P''_n)$, where P''_i is a par-

tition of $[a_i,b_i]$ which refines both P_i and P_i'.) Then

$$L(f,P') \leq L(f,P'') \leq U(f,P'') \leq U(f,P). \quad \blacksquare$$

It follows from Corollary 3-2 that the least upper bound of all lower sums for f is less than or equal to the greatest lower bound of all upper sums for f. A function $f\colon A \to \mathbf{R}$ is called **integrable** on the rectangle A if f is bounded and $\sup\{L(f,P)\} = \inf\{U(f,P)\}$. This common number is then denoted $\int_A f$, and called the **integral** of f over A. Often, the notation $\int_A f(x^1, \ldots, x^n)dx^1 \cdots dx^n$ is used. If $f\colon [a,b] \to \mathbf{R}$, where $a \leq b$, then $\int_a^b f = \int_{[a,b]} f$. A simple but useful criterion for integrability is provided by

3-3 Theorem. *A bounded function $f\colon A \to \mathbf{R}$ is integrable if and only if for every $\varepsilon > 0$ there is a partition P of A such that $U(f,P) - L(f,P) < \varepsilon$.*

Proof. If this condition holds, it is clear that $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ and f is integrable. On the other hand, if f is integrable, so that $\sup\{L(f,P)\} = \inf\{U(f,P)\}$, then for any $\varepsilon > 0$ there are partitions P and P' with $U(f,P) - L(f,P') < \varepsilon$. If P'' refines both P and P', it follows from Lemma 3-1 that $U(f,P'') - L(f,P'') \leq U(f,P) - L(f,P') < \varepsilon$. $\blacksquare$

In the following sections we will characterize the integrable functions and discover a method of computing integrals. For the present we consider two functions, one integrable and one not.

1. Let $f\colon A \to \mathbf{R}$ be a constant function, $f(x) = c$. Then for any partition P and subrectangle S we have $m_S(f) = M_S(f) = c$, so that $L(f,P) = U(f,P) = \Sigma_S c \cdot v(S) = c \cdot v(A)$. Hence $\int_A f = c \cdot v(A)$.

2. Let $f\colon [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

If P is a partition, then every subrectangle S will contain points (x,y) with x rational, and also points (x,y) with x

irrational. Hence $m_S(f) = 0$ and $M_S(f) = 1$, so

$$L(f,P) = \sum_S 0 \cdot v(S) = 0$$

and

$$U(f,P) = \sum_S 1 \cdot v(S) = v([0,1] \times [0,1]) = 1.$$

Therefore f is not integrable.

Problems. **3-1.** Let $f: [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that f is integrable and $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

3-2. Let $f: A \to \mathbf{R}$ be integrable and let $g = f$ except at finitely many points. Show that g is integrable and $\int_A f = \int_A g$.

3-3. Let $f,g: A \to \mathbf{R}$ be integrable.

(a) For any partition P of A and subrectangle S, show that

$$m_S(f) + m_S(g) \le m_S(f+g) \quad \text{and} \quad M_S(f+g) \le M_S(f) + M_S(g)$$

and therefore

$$L(f,P) + L(g,P) \le L(f+g, P) \quad \text{and} \quad U(f+g, P) \le U(f,P) + U(g,P).$$

(b) Show that $f + g$ is integrable and $\int_A f + g = \int_A f + \int_A g$.

(c) For any constant c, show that $\int_A cf = c\int_A f$.

3-4. Let $f: A \to \mathbf{R}$ and let P be a partition of A. Show that f is integrable if and only if for each subrectangle S the function $f|S$, which consists of f restricted to S, is integrable, and that in this case $\int_A f = \sum_S \int_S f|S$.

3-5. Let $f,g: A \to \mathbf{R}$ be integrable and suppose $f \le g$. Show that $\int_A f \le \int_A g$.

3-6. If $f: A \to \mathbf{R}$ is integrable, show that $|f|$ is integrable and $|\int_A f| \le \int_A |f|$.

3-7. Let $f: [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & x \text{ irrational}, \\ 0 & x \text{ rational}, y \text{ irrational}, \\ 1/q & x \text{ rational}, y = p/q \text{ in lowest terms}. \end{cases}$$

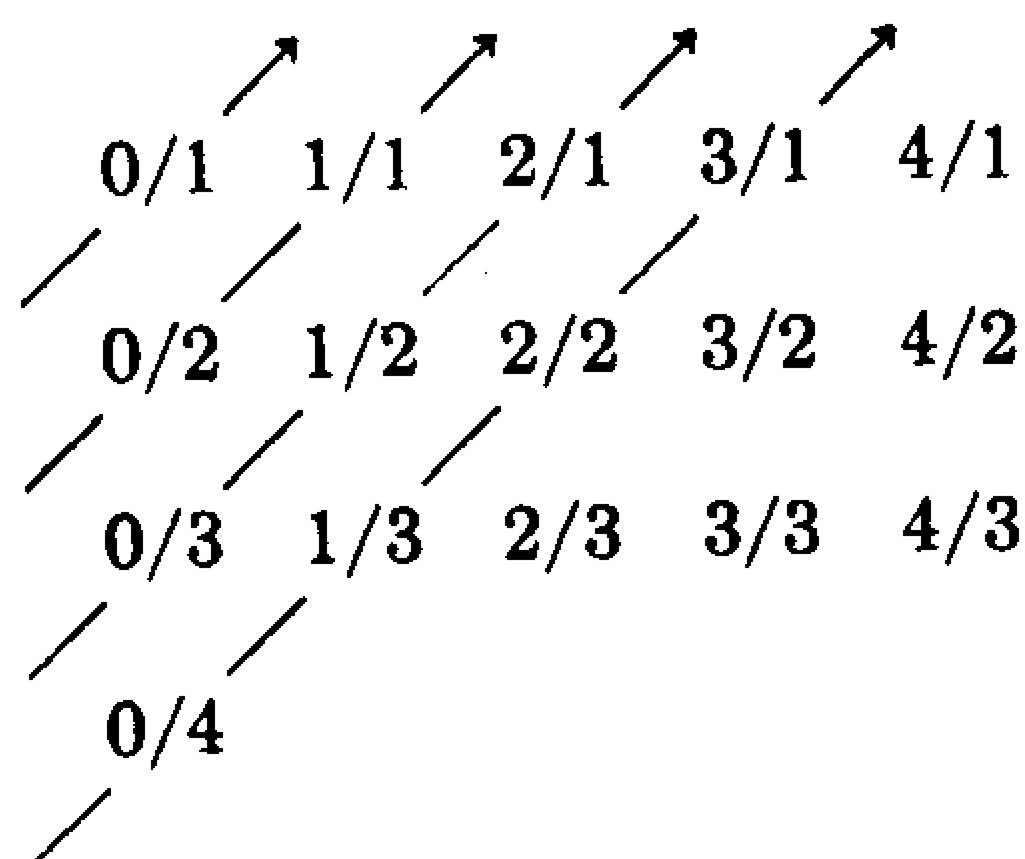
Show that f is integrable and $\int_{[0,1]\times[0,1]} f = 0$.

MEASURE ZERO AND CONTENT ZERO

A subset A of $\mathbf{R}^n$ has (n-dimensional) **measure** 0 if for every $\varepsilon > 0$ there is a cover $\{U_1, U_2, U_3, \ldots\}$ of A by closed rectangles such that $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$. It is obvious (but nevertheless useful to remember) that if A has measure 0 and $B \subset A$, then B has measure 0. The reader may verify that open rectangles may be used instead of closed rectangles in the definition of measure 0.

A set with only finitely many points clearly has measure 0. If A has infinitely many points which can be arranged in a sequence $a_1, a_2, a_3, \ldots$, then A also has measure 0, for if $\varepsilon > 0$, we can choose U_i to be a closed rectangle containing a_i with $v(U_i) < \varepsilon/2^i$. Then $\sum_{i=1}^{\infty} v(U_i) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$.

The set of all rational numbers between 0 and 1 is an important and rather surprising example of an infinite set whose members can be arranged in such a sequence. To see that this is so, list the fractions in the following array in the order indicated by the arrows (deleting repetitions and numbers greater than 1):



An important generalization of this idea can be given.

3-4 *Theorem.* *If* $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ *and each* A_i *has measure* 0, *then* A *has measure* 0.

Proof. Let $\varepsilon > 0$. Since A_i has measure 0, there is a cover $\{U_{i,1}, U_{i,2}, U_{i,3}, \ldots\}$ of A_i by closed rectangles such that $\sum_{j=1}^{\infty} v(U_{i,j}) < \varepsilon/2^i$. Then the collection of all $U_{i,j}$ is a cover

of A. By considering the array

$$\begin{array}{cccc} \nearrow & \nearrow & \nearrow & \\ U_{1,1} & U_{1,2} & U_{1,3} & \cdots \\ \swarrow & \swarrow & \swarrow & \\ U_{2,1} & U_{2,2} & U_{2,3} & \cdots \\ \swarrow & \swarrow & & \\ U_{3,1} & U_{3,2} & U_{3,3} & \cdots \\ \swarrow & & & \end{array}$$

we see that this collection can be arranged in a sequence $V_1, V_2, V_3, \ldots$. Clearly $\sum_{i=1}^{\infty} v(V_i) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$. ▮

A subset A of $\mathbf{R}^n$ has (n-dimensional) **content** 0 if for every $\varepsilon > 0$ there is a *finite* cover $\{U_1, \ldots, U_n\}$ of A by closed rectangles such that $\sum_{i=1}^{n} v(U_i) < \varepsilon$. If A has content 0, then A clearly has measure 0. Again, open rectangles could be used instead of closed rectangles in the definition.

3-5 Theorem. *If $a < b$, then $[a,b] \subset \mathbf{R}$ does* not *have content 0. In fact, if $\{U_1, \ldots, U_n\}$ is a finite cover of $[a,b]$ by closed intervals, then $\sum_{i=1}^{n} v(U_i) \geq b - a$.*

Proof. Clearly we can assume that each $U_i \subset [a,b]$. Let $a = t_0 < t_1 < \ldots < t_k = b$ be all endpoints of all U_i. Then each $v(U_i)$ is the sum of certain $t_j - t_{j-1}$. Moreover, each $[t_{j-1},t_j]$ lies in at least one U_i (namely, any one which contains an interior point of $[t_{j-1},t_j]$), so $\sum_{i=1}^{n} v(U_i) \geq \sum_{j=1}^{k} (t_j - t_{j-1}) = b - a$. ▮

If $a < b$, it is also true that $[a,b]$ does not have measure 0. This follows from

3-6 Theorem. *If A is compact and has measure 0, then A has content 0.*

Proof. Let $\varepsilon > 0$. Since A has measure 0, there is a cover $\{U_1, U_2, \ldots\}$ of A by open rectangles such that $\sum_{i=1}^{\infty} v(U_i)$

$< \varepsilon$. Since A is compact, a finite number $U_1, \ldots, U_n$ of the U_i also cover A and surely $\Sigma_{i=1}^n v(U_i) < \varepsilon$. ∎

The conclusion of Theorem 3-6 is false if A is not compact. For example, let A be the set of rational numbers between 0 and 1; then A has measure 0. Suppose, however, that $\{[a_1,b_1], \ldots, [a_n,b_n]\}$ covers A. Then A is contained in the closed set $[a_1,b_1] \cup \cdots \cup [a_n,b_n]$, and therefore $[0,1] \subset [a_1,b_1] \cup \cdots \cup [a_n,b_n]$. It follows from Theorem 3-5 that $\Sigma_{i=1}^n (b_i - a_i) \geq 1$ for any such cover, and consequently A does not have content 0.

Problems. **3-8.** Prove that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ does not have content 0 if $a_i < b_i$ for each i.

3-9. (a) Show that an unbounded set cannot have content 0.
(b) Give an example of a closed set of measure 0 which does not have content 0.

3-10. (a) If C is a set of content 0, show that the boundary of C has content 0.
(b) Give an example of a bounded set C of measure 0 such that the boundary of C does not have measure 0.

3-11. Let A be the set of Problem 1-18. If $\Sigma_{i=1}^\infty (b_i - a_i) < 1$, show that the boundary of A does not have measure 0.

3-12. Let $f: [a,b] \to \mathbf{R}$ be an increasing function. Show that $\{x: f$ is discontinuous at $x\}$ has measure 0. *Hint:* Use Problem 1-30 to show that $\{x: o(f,x) > 1/n\}$ is finite, for each integer n.

3-13.* (a) Show that the collection of all rectangles $[a_1,b_1] \times \cdots \times [a_n,b_n]$ with all a_i and b_i rational can be arranged in a sequence.
(b) If $A \subset \mathbf{R}^n$ is any set and $\mathcal{O}$ is an open cover of A, show that there is a sequence $U_1, U_2, U_3, \ldots$ of members of $\mathcal{O}$ which also cover A. *Hint:* For each $x \in A$ there is a rectangle $B = [a_1,b_1] \times \cdots \times [a_n,b_n]$ with all a_i and b_i rational such that $x \in B \subset U$ for some $U \in \mathcal{O}$.

INTEGRABLE FUNCTIONS

Recall that $o(f,x)$ denotes the oscillation of f at x.

3-7 Lemma. *Let A be a closed rectangle and let $f: A \to \mathbf{R}$ be a bounded function such that $o(f,x) < \varepsilon$ for all $x \in A$. Then there is a partition P of A with $U(f,P) - L(f,P) < \varepsilon \cdot v(A)$.*

Proof. For each $x \in A$ there is a closed rectangle U_x, containing x in its interior, such that $M_{U_x}(f) - m_{U_x}(f) < \varepsilon$. Since A is compact, a finite number $U_{x_1}, \ldots, U_{x_n}$ of the sets U_x cover A. Let P be a partition for A such that each subrectangle S of P is contained in some U_{x_i}. Then $M_S(f) - m_S(f) < \varepsilon$ for each subrectangle S of P, so that $U(f,P) - L(f,P) = \Sigma_S[M_S(f) - m_S(f)] \cdot v(S) < \varepsilon \cdot v(A)$. ∎

3-8 Theorem. *Let A be a closed rectangle and $f\colon A \to \mathbf{R}$ a bounded function. Let $B = \{x\colon f$ is not continuous at $x\}$. Then f is integrable if and only if B is a set of measure 0.*

Proof. Suppose first that B has measure 0. Let $\varepsilon > 0$ and let $B_\varepsilon = \{x\colon\ o(f,x) \geq \varepsilon\}$. Then $B_\varepsilon \subset B$, so that B_ε has measure 0. Since (Theorem 1-11) B_ε is compact, B_ε has content 0. Thus there is a finite collection $U_1, \ldots, U_n$ of closed rectangles, whose interiors cover B_ε, such that $\Sigma_{i=1}^n v(U_i) < \varepsilon$. Let P be a partition of A such that every subrectangle S of P is in one of two groups (see Figure 3-1):

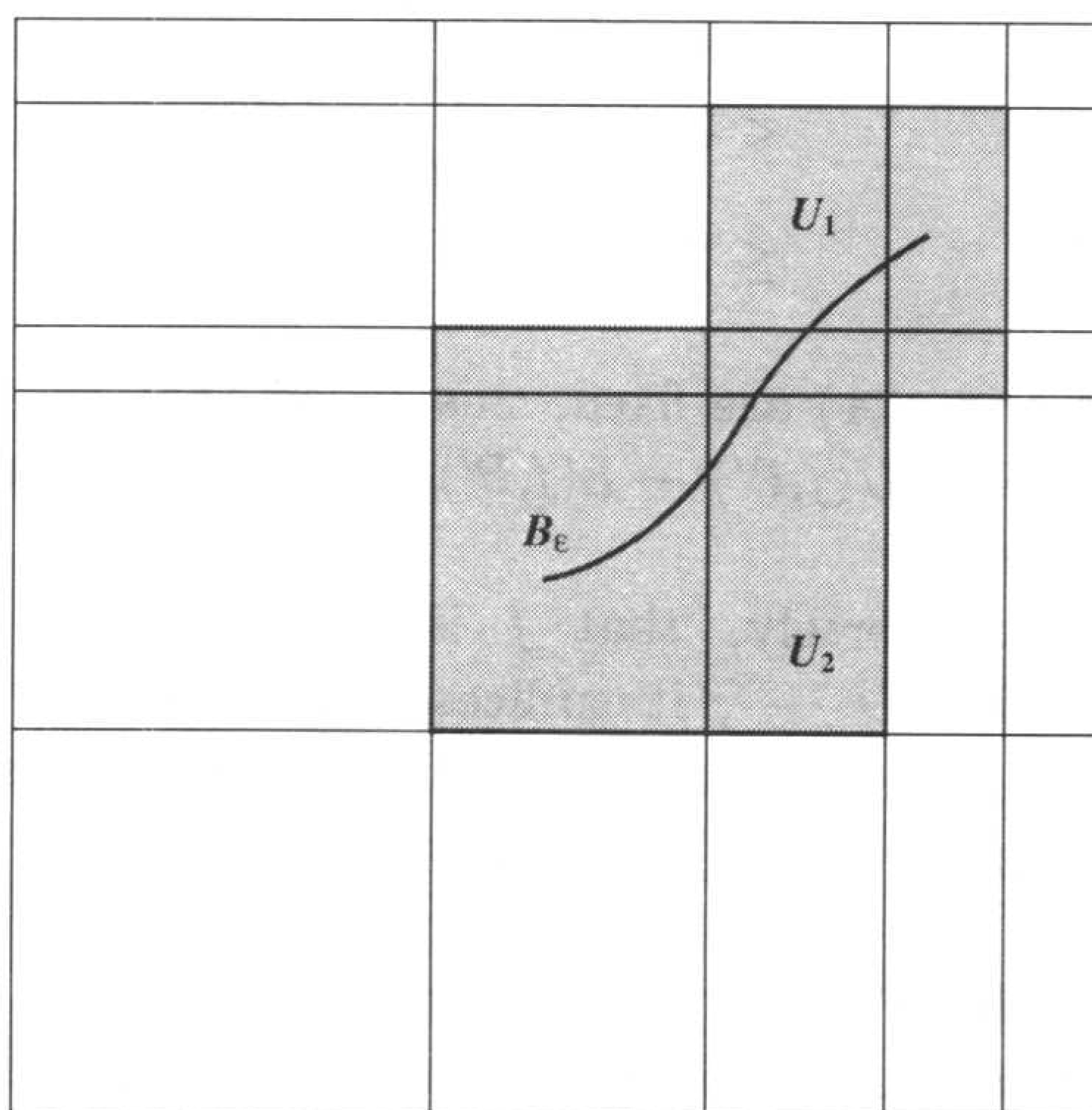


FIGURE 3-1. *The shaded rectangles are in $\mathcal{S}_1$.*

(1) $\mathcal{S}_1$, which consists of subrectangles S, such that $S \subset U_i$ for some i.
(2) $\mathcal{S}_2$, which consists of subrectangles S with $S \cap B_\varepsilon = \emptyset$.

Let $|f(x)| < M$ for $x \in A$. Then $M_S(f) - m_S(f) < 2M$ for every S. Therefore

$$\sum_{S \in \mathcal{S}_1} [M_S(f) - m_S(f)] \cdot v(S) < 2M \sum_{i=1}^{n} v(U_i) < 2M\varepsilon.$$

Now, if $S \in \mathcal{S}_2$, then $o(f,x) < \varepsilon$ for $x \in S$. Lemma 3-7 implies that there is a refinement P' of P such that

$$\sum_{S' \subset S} [M_{S'}(f) - m_{S'}(f)] \cdot v(S') < \varepsilon \cdot v(S)$$

for $S \in \mathcal{S}_2$. Then

$$\begin{aligned} U(f,P') - L(f,P') &= \sum_{S' \subset S \in \mathcal{S}_1} [M_{S'}(f) - m_{S'}(f)] \cdot v(S') \\ &\quad + \sum_{S' \subset S \in \mathcal{S}_2} [M_{S'}(f) - m_{S'}(f)] \cdot v(S') \\ &< 2M\varepsilon + \sum_{S \in \mathcal{S}_2} \varepsilon \cdot v(S) \\ &\leq 2M\varepsilon + \varepsilon \cdot v(A). \end{aligned}$$

Since M and $v(A)$ are fixed, this shows that we can find a partition P' with $U(f,P') - L(f,P')$ as small as desired. Thus f is integrable.

Suppose, conversely, that f is integrable. Since $B = B_1 \cup B_{\frac{1}{2}} \cup B_{\frac{1}{3}} \cup \cdots$, it suffices (Theorem 3-4) to prove that each $B_{1/n}$ has measure 0. In fact we will show that each $B_{1/n}$ has content 0 (since $B_{1/n}$ is compact, this is actually equivalent).

If $\varepsilon > 0$, let P be a partition of A such that $U(f,P) - L(f,P) < \varepsilon/n$. Let $\mathcal{S}$ be the collection of subrectangles S of P which intersect $B_{1/n}$. Then $\mathcal{S}$ is a cover of $B_{1/n}$. Now if

$S \in \mathcal{S}$, then $M_S(f) - m_S(f) \geq 1/n$. Thus

$$\begin{aligned}\frac{1}{n} \cdot \sum_{S \in \mathcal{S}} v(S) &\leq \sum_{S \in \mathcal{S}} [M_S(f) - m_S(f)] \cdot v(S) \\ &\leq \sum_{S} [M_S(f) - m_S(f)] \cdot v(S) \\ &< \frac{\varepsilon}{n},\end{aligned}$$

and consequently $\Sigma_{S \in \mathcal{S}} v(S) < \varepsilon$. ▌

We have thus far dealt only with the integrals of functions over rectangles. Integrals over other sets are easily reduced to this type. If $C \subset \mathbf{R}^n$, the **characteristic function** χ_C of C is defined by

$$\chi_C(x) = \begin{cases} 0 & x \notin C, \\ 1 & x \in C. \end{cases}$$

If $C \subset A$ for some closed rectangle A and $f: A \to \mathbf{R}$ is bounded, then $\int_C f$ is defined as $\int_A f \cdot \chi_C$, provided $f \cdot \chi_C$ is integrable. This certainly occurs (Problem 3-14) if f and χ_C are integrable.

3-9 Theorem. *The function $\chi_C: A \to \mathbf{R}$ is integrable if and only if the boundary of C has measure 0 (and hence content 0).*

Proof. If x is in the interior of C, then there is an open rectangle U with $x \in U \subset C$. Thus $\chi_C = 1$ on U and χ_C is clearly continuous at x. Similarly, if x is in the exterior of C, there is an open rectangle U with $x \in U \subset \mathbf{R}^n - C$. Hence $\chi_C = 0$ on U and χ_C is continuous at x. Finally, if x is in the boundary of C, then for every open rectangle U containing x, there is $y_1 \in U \cap C$, so that $\chi_C(y_1) = 1$ and there is $y_2 \in U \cap (\mathbf{R}^n - C)$, so that $\chi_C(y_2) = 0$. Hence χ_C is not continuous at x. Thus $\{x: \chi_C \text{ is not continuous at } x\} =$ boundary C, and the result follows from Theorem 3-8. ▌

A bounded set C whose boundary has measure 0 is called **Jordan-measurable.** The integral $\int_C 1$ is called the (n-dimensional) **content** of C, or the (n-dimensional) **volume** of C. Naturally one-dimensional volume is often called **length,** and two-dimensional volume, **area.**

Problem 3-11 shows that even an open set C may not be Jordan-measurable, so that $\int_C f$ is not necessarily defined even if C is open and f is continuous. This unhappy state of affairs will be rectified soon.

Problems. **3-14.** Show that if $f,g: A \to \mathbf{R}$ are integrable, so is $f \cdot g$.

3-15. Show that if C has content 0, then $C \subset A$ for some closed rectangle A and C is Jordan-measurable and $\int_A \chi_C = 0$.

3-16. Give an example of a bounded set C of measure 0 such that $\int_A \chi_C$ does not exist.

3-17. If C is a bounded set of measure 0 and $\int_A \chi_C$ exists, show that $\int_A \chi_C = 0$. *Hint:* Show that $L(f,P) = 0$ for all partitions P. Use Problem 3-8.

3-18. If $f: A \to \mathbf{R}$ is non-negative and $\int_A f = 0$, show that $\{x: f(x) \neq 0\}$ has measure 0. *Hint:* Prove that $\{x: f(x) > 1/n\}$ has content 0.

3-19. Let U be the open set of Problem 3-11. Show that if $f = \chi_U$ except on a set of measure 0, then f is not integrable on $[0,1]$.

3-20. Show that an increasing function $f: [a,b] \to \mathbf{R}$ is integrable on $[a,b]$.

3-21. If A is a closed rectangle, show that $C \subset A$ is Jordan-measurable if and only if for every $\varepsilon > 0$ there is a partition P of A such that $\sum_{S \in \mathcal{S}_1} v(S) - \sum_{S \in \mathcal{S}_2} v(S) < \varepsilon$, where $\mathcal{S}_1$ consists of all subrectangles intersecting C and $\mathcal{S}_2$ all subrectangles contained in C.

3-22.* If A is a Jordan-measurable set and $\varepsilon > 0$, show that there is a compact Jordan-measurable set $C \subset A$ such that $\int_{A-C} 1 < \varepsilon$.

FUBINI'S THEOREM

The problem of calculating integrals is solved, in some sense, by Theorem 3-10, which reduces the computation of integrals over a closed rectangle in $\mathbf{R}^n$, $n > 1$, to the computation of integrals over closed intervals in $\mathbf{R}$. Of sufficient importance to deserve a special designation, this theorem is usually referred to as Fubini's theorem, although it is more or less a

special case of a theorem proved by Fubini long after Theorem 3-10 was known.

The idea behind the theorem is best illustrated (Figure 3-2) for a positive continuous function $f: [a,b] \times [c,d] \to \mathbf{R}$. Let $t_0, \ldots, t_n$ be a partition of $[a,b]$ and divide $[a,b] \times [c,d]$ into n strips by means of the line segments $\{t_i\} \times [c,d]$. If g_x is defined by $g_x(y) = f(x,y)$, then the area of the region under the graph of f and above $\{x\} \times [c,d]$ is

$$\int_c^d g_x = \int_c^d f(x,y)dy.$$

The volume of the region under the graph of f and above $[t_{i-1},t_i] \times [c,d]$ is therefore approximately equal to $(t_i - t_{i-1}) \cdot \int_c^d f(x,y)dy$, for any $x \in [t_{i-1},t_i]$. Thus

$$\int_{[a,b] \times [c,d]} f = \sum_{i=1}^{n} \int_{[t_{i-1},t_i] \times [c,d]} f$$

is approximately $\Sigma_{i=1}^n (t_i - t_{i-1}) \cdot \int_c^d f(x_i,y)dy$, with x_i in

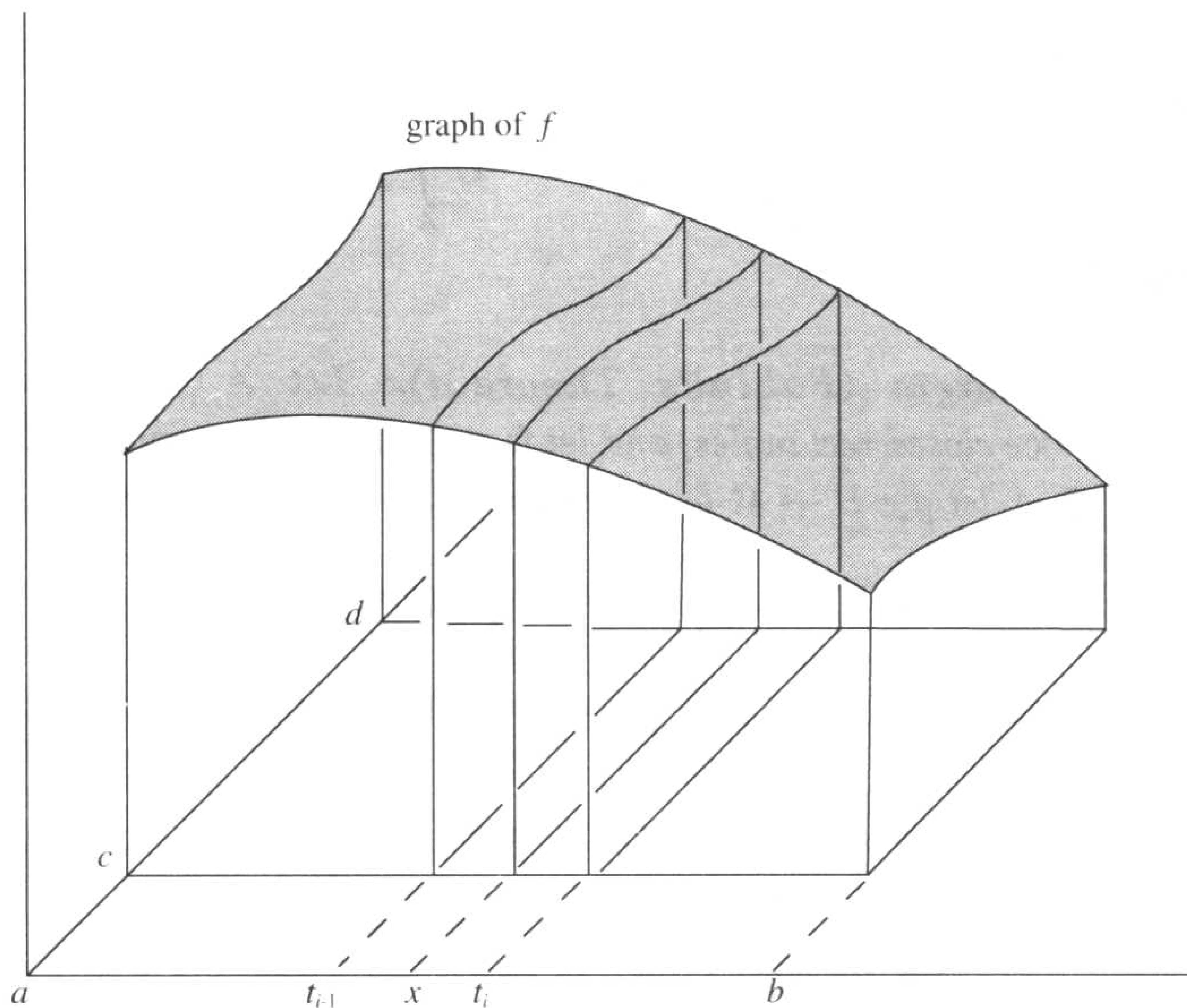


FIGURE 3-2

$[t_{i-1},t_i]$. On the other hand, sums similar to these appear in the definition of $\int_a^b(\int_c^d f(x,y)dy)dx$. Thus, if h is defined by $h(x) = \int_c^d g_x = \int_c^d f(x,y)dy$, it is reasonable to hope that h is integrable on $[a,b]$ and that

$$\int\limits_{[a,b]\times[c,d]} f = \int\limits_a^b h = \int\limits_a^b \left(\int\limits_c^d f(x,y)dy\right) dx.$$

This will indeed turn out to be true when f is continuous, but in the general case difficulties arise. Suppose, for example, that the set of discontinuities of f is $\{x_0\} \times [c,d]$ for some $x_0 \in [a,b]$. Then f is integrable on $[a,b] \times [c,d]$ but $h(x_0) = \int_c^d f(x_0,y)dy$ may not even be defined. The statement of Fubini's theorem therefore looks a little strange, and will be followed by remarks about various special cases where simpler statements are possible.

We will need one bit of terminology. If $f\colon A \to \mathbf{R}$ is a bounded function on a closed rectangle, then, whether or not f is integrable, the least upper bound of all lower sums, and the greatest lower bound of all upper sums, both exist. They are called the **lower** and **upper integrals** of f on A, and denoted

$$\mathbf{L}\int\limits_A f \qquad \text{and} \qquad \mathbf{U}\int\limits_A f,$$

respectively.

3-10 Theorem (Fubini's Theorem). *Let $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^m$ be closed rectangles, and let $f\colon A \times B \to \mathbf{R}$ be integrable. For $x \in A$ let $g_x\colon B \to \mathbf{R}$ be defined by $g_x(y) = f(x,y)$ and let*

$$\mathfrak{L}(x) = \mathbf{L}\int\limits_B g_x = \mathbf{L}\int\limits_B f(x,y)dy,$$
$$\mathfrak{U}(x) = \mathbf{U}\int\limits_B g_x = \mathbf{U}\int\limits_B f(x,y)dy.$$

Then $\mathfrak{L}$ and $\mathfrak{U}$ are integrable on A and

$$\int\limits_{A\times B} f = \int\limits_A \mathfrak{L} = \int\limits_A \left(\mathbf{L}\int\limits_B f(x,y)dy\right) dx,$$
$$\int\limits_{A\times B} f = \int\limits_A \mathfrak{U} = \int\limits_A \left(\mathbf{U}\int\limits_B f(x,y)dy\right) dx.$$

(The integrals on the right side are called **iterated integrals** for f.)

Proof. Let P_A be a partition of A and P_B a partition of B. Together they give a partition P of $A \times B$ for which any subrectangle S is of the form $S_A \times S_B$, where S_A is a subrectangle of the partition P_A, and S_B is a subrectangle of the partition P_B. Thus

$$\begin{aligned} L(f,P) = \sum_S m_S(f) \cdot v(S) &= \sum_{S_A,S_B} m_{S_A \times S_B}(f) \cdot v(S_A \times S_B) \\ &= \sum_{S_A} \Big(\sum_{S_B} m_{S_A \times S_B}(f) \cdot v(S_B) \Big) \cdot v(S_A). \end{aligned}$$

Now, if $x \in S_A$, then clearly $m_{S_A \times S_B}(f) \leq m_{S_B}(g_x)$. Consequently, for $x \in S_A$ we have

$$\sum_{S_B} m_{S_A \times S_B}(f) \cdot v(S_B) \leq \sum_{S_B} m_{S_B}(g_x) \cdot v(S_B) \leq \mathbf{L} \int_B g_x = \mathfrak{L}(x).$$

Therefore

$$\sum_{S_A} \Big(\sum_{S_B} m_{S_A \times S_B}(f) \cdot v(S_B) \Big) \cdot v(S_A) \leq L(\mathfrak{L},P_A).$$

We thus obtain

$$L(f,P) \leq L(\mathfrak{L},P_A) \leq U(\mathfrak{L},P_A) \leq U(\mathfrak{U},P_A) \leq U(f,P),$$

where the proof of the last inequality is entirely analogous to the proof of the first. Since f is integrable, $\sup\{L(f,P)\} = \inf\{U(f,P)\} = \int_{A \times B} f$. Hence

$$\sup\{L(\mathfrak{L},P_A)\} = \inf\{U(\mathfrak{L},P_A)\} = \int_{A \times B} f.$$

In other words, $\mathfrak{L}$ is integrable on A and $\int_{A \times B} f = \int_A \mathfrak{L}$. The assertion for $\mathfrak{U}$ follows similarly from the inequalities

$$L(f,P) \leq L(\mathfrak{L},P_A) \leq L(\mathfrak{U},P_A) \leq U(\mathfrak{U},P_A) \leq U(f,P). \quad \blacksquare$$

Remarks. 1. A similar proof shows that

$$\int_{A \times B} f = \int_B \Big(\mathbf{L} \int_A f(x,y)dx \Big) dy = \int_B \Big(\mathbf{U} \int_A f(x,y)dx \Big) dy.$$

These integrals are called *iterated integrals* for f in the reverse order from those of the theorem. As several problems show, the possibility of interchanging the orders of iterated integrals has many consequences.

2. In practice it is often the case that each g_x is integrable, so that $\int_{A\times B} f = \int_A\left(\int_B f(x,y)dy\right)dx$. This certainly occurs if f is continuous.

3. The worst irregularity commonly encountered is that g_x is not integrable for a finite number of $x \in A$. In this case $\mathfrak{L}(x) = \int_B f(x,y)dy$ for all but these finitely many x. Since $\int_A \mathfrak{L}$ remains unchanged if $\mathfrak{L}$ is redefined at a finite number of points, we can still write $\int_{A\times B} f = \int_A\left(\int_B f(x,y)dy\right)dx$, provided that $\int_B f(x,y)dy$ is defined arbitrarily, say as 0, when it does not exist.

4. There are cases when this will not work and Theorem 3-10 must be used as stated. Let $f\colon [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational and } y \text{ is irrational,} \\ 1 - 1/q & \text{if } x = p/q \text{ in lowest terms and } y \text{ is rational.} \end{cases}$$

Then f is integrable and $\int_{[0,1]\times[0,1]} f = 1$. Now $\int_0^1 f(x,y)dy = 1$ if x is irrational, and does not exist if x is rational. Therefore h is not integrable if $h(x) = \int_0^1 f(x,y)dy$ is set equal to 0 when the integral does not exist.

5. If $A = [a_1,b_1] \times \cdots \times [a_n,b_n]$ and $f\colon A \to \mathbf{R}$ is sufficiently nice, we can apply Fubini's theorem repeatedly to obtain

$$\int_A f = \int_{a_n}^{b_n}\left(\cdots\left(\int_{a_1}^{b_1} f(x^1, \ldots, x^n)dx^1\right)\cdots\right)dx^n.$$

6. If $C \subset A \times B$, Fubini's theorem can be used to evaluate $\int_C f$, since this is by definition $\int_{A\times B} \chi_C f$. Suppose, for example, that

$$C = [-1,1] \times [-1,1] - \{(x,y)\colon |(x,y)| < 1\}.$$

Then

$$\int_C f = \int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)\cdot\chi_C(x,y)dy\right)dx.$$

Now

$$\chi_C(x,y) = \begin{cases} 1 & \text{if } y > \sqrt{1-x^2} \text{ or } y < -\sqrt{1-x^2}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\int_{-1}^{1} f(x,y) \cdot \chi_C(x,y)dy = \int_{-1}^{-\sqrt{1-x^2}} f(x,y)dy + \int_{\sqrt{1-x^2}}^{1} f(x,y)dy.$$

In general, if $C \subset A \times B$, the main difficulty in deriving expressions for $\int_C f$ will be determining $C \cap (\{x\} \times B)$ for $x \in A$. If $C \cap (A \times \{y\})$ for $y \in B$ is easier to determine, one should use the iterated integral

$$\int_C f = \int_B \left(\int_A f(x,y) \cdot \chi_C(x,y)dx \right) dy.$$

Problems. **3-23.** Let $C \subset A \times B$ be a set of content 0. Let $A' \subset A$ be the set of all $x \in A$ such that $\{y \in B: (x,y) \in C\}$ is *not* of content 0. Show that A' is a set of measure 0. *Hint:* χ_C is integrable and $\int_{A \times B} \chi_C = \int_A \mathfrak{U} = \int_A \mathfrak{L}$, so $\int_A \mathfrak{U} - \mathfrak{L} = 0$.

3-24. Let $C \subset [0,1] \times [0,1]$ be the union of all $\{p/q\} \times [0, 1/q]$, where p/q is a rational number in $[0,1]$ written in lowest terms. Use C to show that the word "measure" in Problem 3-23 cannot be replaced by "content."

3-25. Use induction on n to show that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ is not a set of measure 0 (or content 0) if $a_i < b_i$ for each i.

3-26. Let $f: [a,b] \to \mathbf{R}$ be integrable and non-negative and let $A_f = \{(x,y): a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$. Show that A_f is Jordan-measurable and has area $\int_a^b f$.

3-27. If $f: [a,b] \times [a,b] \to \mathbf{R}$ is continuous, show that

$$\int_a^b \int_a^y f(x,y)dx\, dy = \int_a^b \int_x^b f(x,y)dy\, dx.$$

Hint: Compute $\int_C f$ in two different ways for a suitable set $C \subset [a,b] \times [a,b]$.

3-28.* Use Fubini's theorem to give an easy proof that $D_{1,2}f = D_{2,1}f$ if these are continuous. *Hint:* If $D_{1,2}f(a) - D_{2,1}f(a) > 0$, there is a rectangle A containing a such that $D_{1,2}f - D_{2,1}f > 0$ on A.

3-29. Use Fubini's theorem to derive an expression for the volume of a set of $\mathbf{R}^3$ obtained by revolving a Jordan-measurable set in the yz-plane about the z-axis.

3-30. Let C be the set in Problem 1-17. Show that

$$\int_{[0,1]}\left(\int_{[0,1]}\chi_C(x,y)dx\right)dy = \int_{[0,1]}\left(\int_{[0,1]}\chi_C(y,x)dy\right)dx = 0$$

but that $\int_{[0,1]\times[0,1]}\chi_C$ does not exist.

3-31. If $A = [a_1,b_1] \times \cdots \times [a_n,b_n]$ and $f\colon A \to \mathbf{R}$ is continuous, define $F\colon A \to \mathbf{R}$ by

$$F(x) = \int_{[a_1,x^1]\times\cdots\times[a_n,x^n]} f.$$

What is $D_iF(x)$, for x in the interior of A?

3-32.* Let $f\colon [a,b] \times [c,d] \to \mathbf{R}$ be continuous and suppose D_2f is continuous. Define $F(y) = \int_a^b f(x,y)dx$. Prove *Leibnitz's rule:* $F'(y) = \int_a^b D_2f(x,y)dx$. *Hint:* $F(y) = \int_a^b f(x,y)dx = \int_a^b(\int_c^y D_2f(x,y)dy + f(x,c))dx$. (The proof will show that continuity of D_2f may be replaced by considerably weaker hypotheses.)

3-33. If $f\colon [a,b] \times [c,d] \to \mathbf{R}$ is continuous and D_2f is continuous, define $F(x,y) = \int_a^x f(t,y)dt$.

(a) Find D_1F and D_2F.

(b) If $G(x) = \int_a^{g(x)} f(t,x)dt$, find $G'(x)$.

3-34.* Let $g_1,g_2\colon \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable and suppose $D_1g_2 = D_2g_1$. As in Problem 2-21, let

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

Show that $D_1f(x,y) = g_1(x,y)$.

3-35.* (*a*) Let $g\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation of one of the following types:

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases}$$

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$$

$$\begin{cases} g(e_k) = e_k & k \neq i, j \\ g(e_i) = e_j \\ g(e_j) = e_i. \end{cases}$$

If U is a rectangle, show that the volume of $g(U)$ is $|\det g| \cdot v(U)$.

(b) Prove that $|\det g| \cdot v(U)$ is the volume of $g(U)$ for any linear transformation $g\colon \mathbf{R}^n \to \mathbf{R}^n$. *Hint:* If $\det g \neq 0$, then g is the composition of linear transformations of the type considered in (a).

3-36. (Cavalieri's principle). Let A and B be Jordan-measurable subsets of $\mathbf{R}^3$. Let $A_c = \{(x,y)\colon (x,y,c) \in A\}$ and define B_c similarly. Suppose each A_c and B_c are Jordan-measurable and have the same area. Show that A and B have the same volume.

PARTITIONS OF UNITY

In this section we introduce a tool of extreme importance in the theory of integration.

3-11 Theorem. *Let $A \subset \mathbf{R}^n$ and let $\mathcal{O}$ be an open cover of A. Then there is a collection Φ of C^∞ functions φ defined in an open set containing A, with the following properties:*

(1) *For each $x \in A$ we have $0 \leq \varphi(x) \leq 1$.*
(2) *For each $x \in A$ there is an open set V containing x such that all but finitely many $\varphi \in \Phi$ are 0 on V.*
(3) *For each $x \in A$ we have $\Sigma_{\varphi \in \Phi} \varphi(x) = 1$* (by (2) for each x this sum is finite in some open set containing x).
(4) *For each $\varphi \in \Phi$ there is an open set U in $\mathcal{O}$ such that $\varphi = 0$ outside of some closed set contained in U.*

(A collection Φ satisfying (1) to (3) is called a C^∞ **partition of unity** for A. If Φ also satisfies (4), it is said to be **subordinate** to the cover $\mathcal{O}$. In this chapter we will only use continuity of the functions φ.)

Proof. *Case 1. A is compact.*

Then a finite number $U_1, \ldots, U_n$ of open sets in $\mathcal{O}$ cover A. It clearly suffices to construct a partition of unity subordinate to the cover $\{U_1, \ldots, U_n\}$. We will first find compact sets $D_i \subset U_i$ whose interiors cover A. The sets D_i are constructed inductively as follows. Suppose that $D_1, \ldots, D_k$ have been chosen so that $\{$interior $D_1, \ldots,$ interior D_k, $U_{k+1}, \ldots, U_n\}$ covers A. Let

$$C_{k+1} = A - (\text{int } D_1 \cup \cdots \cup \text{int } D_k \cup U_{k+2} \cup \cdots \cup U_n).$$

Then $C_{k+1} \subset U_{k+1}$ is compact. Hence (Problem 1-22) we can find a compact set D_{k+1} such that

$$C_{k+1} \subset \text{interior } D_{k+1} \quad \text{and} \quad D_{k+1} \subset U_{k+1}.$$

Having constructed the sets $D_1, \ldots, D_n$, let ψ_i be a nonnegative C^∞ function which is positive on D_i and 0 outside of some closed set contained in U_i (Problem 2-26). Since

$\{D_1, \ldots, D_n\}$ covers A, we have $\psi_1(x) + \cdots + \psi_n(x) > 0$ for all x in some open set U containing A. On U we can define

$$\varphi_i(x) = \frac{\psi_i(x)}{\psi_1(x) + \cdots + \psi_n(x)}.$$

If $f\colon U \to [0,1]$ is a C^∞ function which is 1 on A and 0 outside of some closed set in U, then $\Phi = \{f \cdot \varphi_1, \ldots, f \cdot \varphi_n\}$ is the desired partition of unity.

Case 2. $A = A_1 \cup A_2 \cup A_3 \cup \cdots$, *where each* A_i *is compact and* $A_i \subset$ *interior* A_{i+1}.

For each i let $\mathcal{O}_i$ consist of all $U \cap$ (interior $A_{i+1} - A_{i-2}$) for U in $\mathcal{O}$. Then $\mathcal{O}_i$ is an open cover of the compact set $B_i = A_i -$ interior A_{i-1}. By case 1 there is a partition of unity Φ_i for B_i, subordinate to $\mathcal{O}_i$. For each $x \in A$ the sum

$$\sigma(x) = \sum_{\varphi \in \Phi_i,\ \text{all}\ i} \varphi(x)$$

is a finite sum in some open set containing x, since if $x \in A_i$ we have $\varphi(x) = 0$ for $\varphi \in \Phi_j$ with $j \geq i + 2$. For each φ in each Φ_i, define $\varphi'(x) = \varphi(x)/\sigma(x)$. The collection of all φ' is the desired partition of unity.

Case 3. A *is open.*

Let $A_i =$

$$\{x \in A\colon |x| \leq i \text{ and distance from } x \text{ to boundary } A \geq 1/i\},$$

and apply case 2.

Case 4. A *is arbitrary.*

Let B be the union of all U in $\mathcal{O}$. By case 3 there is a partition of unity for B; this is also a partition of unity for A. ∎

An important consequence of condition (2) of the theorem should be noted. Let $C \subset A$ be compact. For each $x \in C$ there is an open set V_x containing x such that only finitely many $\varphi \in \Phi$ are not 0 on V_x. Since C is compact, finitely many such V_x cover C. Thus only finitely many $\varphi \in \Phi$ are not 0 on C.

One important application of partitions of unity will illustrate their main role—piecing together results obtained locally.

An open cover $\mathcal{O}$ of an open set $A \subset \mathbf{R}^n$ is **admissible** if each $U \in \mathcal{O}$ is contained in A. If Φ is subordinate to $\mathcal{O}$, $f: A \to \mathbf{R}$ is bounded in some open set around each point of A, and $\{x: f \text{ is discontinuous at } x\}$ has measure 0, then each $\int_A \varphi \cdot |f|$ exists. We define f to be **integrable** (in the extended sense) if $\sum_{\varphi \in \Phi} \int_A \varphi \cdot |f|$ converges (the proof of Theorem 3-11 shows that the φ's may be arranged in a sequence). This implies convergence of $\sum_{\varphi \in \Phi} \left| \int_A \varphi \cdot f \right|$, and hence absolute convergence of $\sum_{\varphi \in \Phi} \int_A \varphi \cdot f$, which we define to be $\int_A f$. These definitions do not depend on $\mathcal{O}$ or Φ (but see Problem 3-38).

3-12 Theorem.

(1) *If Ψ is another partition of unity, subordinate to an admissible cover $\mathcal{O}'$ of A, then $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$ also converges, and*

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\psi \in \Psi} \int_A \psi \cdot f.$$

(2) *If A and f are bounded, then f is integrable in the extended sense.*

(3) *If A is Jordan-measurable and f is bounded, then this definition of $\int_A f$ agrees with the old one.*

Proof

(1) Since $\varphi \cdot f = 0$ except on some compact set C, and there are only finitely many ψ which are non-zero on C, we can write

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\varphi \in \Phi} \int_A \sum_{\psi \in \Psi} \psi \cdot \varphi \cdot f = \sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \int_A \psi \cdot \varphi \cdot f.$$

This result, applied to $|f|$, shows the convergence of $\sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \int_A \psi \cdot \varphi \cdot |f|$, and hence of $\sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \left| \int_A \psi \cdot \varphi \cdot f \right|$. This absolute convergence justifies interchanging the order of summation in the above equation; the resulting double sum clearly equals $\sum_{\psi \in \Psi} \int_A \psi \cdot f$. Finally, this result applied to $|f|$ proves convergence of $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$.

(2) If A is contained in the closed rectangle B and $|f(x)| \leq M$ for $x \in A$, and $F \subset \Phi$ is finite, then

$$\sum_{\varphi \in F} \int_A \varphi \cdot |f| \leq \sum_{\varphi \in F} M \int_A \varphi = M \int_A \sum_{\varphi \in F} \varphi \leq Mv(B),$$

since $\Sigma_{\varphi \in F}\, \varphi \leq 1$ on A.

(3) If $\varepsilon > 0$ there is (Problem 3-22) a compact Jordan-measurable $C \subset A$ such that $\int_{A-C} 1 < \varepsilon$. There are only finitely many $\varphi \in \Phi$ which are non-zero on C. If $F \subset \Phi$ is any finite collection which includes these, and $\int_A f$ has its old meaning, then

$$\begin{aligned} \left| \int_A f - \sum_{\varphi \in F} \int_A \varphi \cdot f \right| &\leq \int_A \left| f - \sum_{\varphi \in F} \varphi \cdot f \right| \\ &\leq M \int_A \left(1 - \sum_{\varphi \in F} \varphi\right) \\ &= M \int_A \sum_{\varphi \in \Phi - F} \varphi \leq M \int_{A-C} 1 \leq M\varepsilon. \quad ∎ \end{aligned}$$

Problems. **3-37.** (a) Suppose that $f\colon (0,1) \to \mathbf{R}$ is a non-negative continuous function. Show that $\int_{(0,1)} f$ exists if and only if $\lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} f$ exists.

(b) Let $A_n = [1 - 1/2^n, 1 - 1/2^{n+1}]$. Suppose that $f\colon (0,1) \to \mathbf{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f(x) = 0$ for $x \notin$ any A_n. Show that $\int_{(0,1)} f$ does not exist, but $\lim_{\varepsilon \to 0} \int_{(\varepsilon, 1-\varepsilon)} f = \log 2$.

3-38. Let A_n be a closed set contained in $(n, n+1)$. Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ for $x \notin$ any A_n. Find two partitions of unity Φ and Ψ such that $\Sigma_{\varphi \in \Phi} \int_{\mathbf{R}} \varphi \cdot f$ and $\Sigma_{\psi \in \Psi} \int_{\mathbf{R}} \psi \cdot f$ converge absolutely to different values.

CHANGE OF VARIABLE

If $g\colon [a,b] \to \mathbf{R}$ is continuously differentiable and $f\colon \mathbf{R} \to \mathbf{R}$ is continuous, then, as is well known,

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'.$$

The proof is very simple: if $F' = f$, then $(F \circ g)' = (f \circ g) \cdot g'$; thus the left side is $F(g(b)) - F(g(a))$, while the right side is $F \circ g(b) - F \circ g(a) = F(g(b)) - F(g(a))$.

We leave it to the reader to show that if g is 1-1, then the above formula can be written

$$\int_{g((a,b))} f = \int_{(a,b)} f \circ g \cdot |g'|.$$

(Consider separately the cases where g is increasing and where g is decreasing.) The generalization of this formula to higher dimensions is by no means so trivial.

3-13 Theorem. *Let $A \subset \mathbf{R}^n$ be an open set and $g\colon A \to \mathbf{R}^n$ a 1-1, continuously differentiable function such that det $g'(x) \neq 0$ for all $x \in A$. If $f\colon g(A) \to \mathbf{R}$ is integrable, then*

$$\int_{g(A)} f = \int_A (f \circ g)|\det g'|.$$

Proof. We begin with some important reductions.

1. Suppose there is an admissible cover $\mathcal{O}$ for A such that for each $U \in \mathcal{O}$ and any integrable f we have

$$\int_{g(U)} f = \int_U (f \circ g)|\det g'|.$$

Then the theorem is true for all of A. (Since g is automatically 1-1 in an open set around each point, it is not surprising that this is the only part of the proof using the fact that g is 1-1 on all of A.)

Proof of (1). The collection of all $g(U)$ is an open cover of $g(A)$. Let Φ be a partition of unity subordinate to this cover. If $\varphi = 0$ outside of $g(U)$, then, since g is 1-1, we have $(\varphi \cdot f) \circ g$

$= 0$ outside of U. Therefore the equation

$$\int_{g(U)} \varphi \cdot f = \int_U [(\varphi \cdot f) \circ g]|\det g'|.$$

can be written

$$\int_{g(A)} \varphi \cdot f = \int_A [(\varphi \cdot f) \circ g]|\det g'|.$$

Hence

$$\begin{aligned}\int_{g(A)} f = \sum_{\varphi \in \Phi} \int_{g(A)} \varphi \cdot f &= \sum_{\varphi \in \Phi} \int_A [(\varphi \cdot f) \circ g]|\det g'| \\ &= \sum_{\varphi \in \Phi} \int_A (\varphi \circ g)(f \circ g)|\det g'| \\ &= \int_A (f \circ g)|\det g'|.\end{aligned}$$

Remark. The theorem also follows from the assumption that

$$\int_V f = \int_{g^{-1}(V)} (f \circ g)|\det g'|$$

for V in some admissible cover of $g(A)$. This follows from (1) applied to g^{-1}.

2. It suffices to prove the theorem for the function $f = 1$.

Proof of (2). If the theorem holds for $f = 1$, it holds for constant functions. Let V be a rectangle in $g(A)$ and P a partition of V. For each subrectangle S of P let f_S be the constant function $m_S(f)$. Then

$$\begin{aligned}L(f,P) &= \sum_S m_S(f) \cdot v(S) = \sum_S \int_{\text{int } S} f_S \\ &= \sum_S \int_{g^{-1}(\text{int } S)} (f_S \circ g)|\det g' \leq \sum_S \int_{g^{-1}(\text{int } S)} (f \circ g)|\det g'| \\ &\leq \int_{g^{-1}(V)} (f \circ g)|\det g'|.\end{aligned}$$

Since $\int_V f$ is the least upper bound of all $L(f,P)$, this proves that $\int_V f \leq \int_{g^{-1}(V)} (f \circ g)|\det g'|$. A similar argument, letting $f_S = M_S(f)$, shows that $\int_V f \geq \int_{g^{-1}(V)} (f \circ g)|\det g'|$. The result now follows from the above Remark.

3. If the theorem is true for $g\colon A \to \mathbf{R}^n$ and for $h\colon B \to \mathbf{R}^n$, where $g(A) \subset B$, then it is true for $h \circ g\colon A \to \mathbf{R}^n$.

Proof of (3).

$$\int_{h \circ g(A)} f = \int_{h(g(A))} f = \int_{g(A)} (f \circ h)|\det h'|$$
$$= \int_A [(f \circ h) \circ g] \cdot [|\det h'| \circ g] \cdot |\det g'|$$
$$= \int_A f \circ (h \circ g)|\det (h \circ g)'|.$$

4. The theorem is true if g is a linear transformation.

Proof of (4). By (1) and (2) it suffices to show for any open rectangle U that

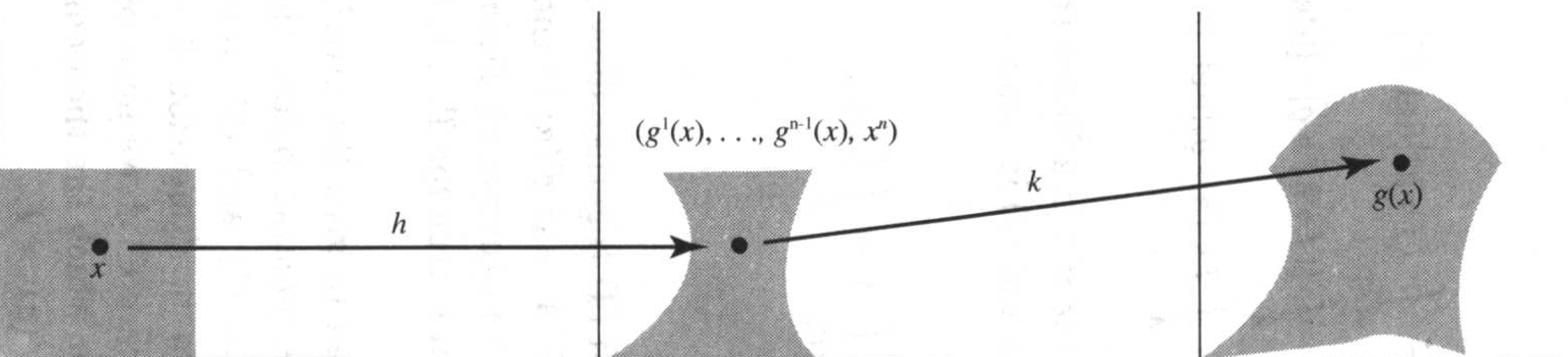
$$\int_{g(U)} 1 = \int_U |\det g'|.$$

This is Problem 3-35.

Observations (3) and (4) together show that we may assume for any particular $a \in A$ that $g'(a)$ is the identity matrix: in fact, if T is the linear transformation $Dg(a)$, then $(T^{-1} \circ g)'(a) = I$; since the theorem is true for T, if it is true for $T^{-1} \circ g$ it will be true for g.

We are now prepared to give the proof, which preceeds by induction on n. The remarks before the statement of the theorem, together with (1) and (2), prove the case $n = 1$. Assuming the theorem in dimension $n - 1$, we prove it in dimension n. For each $a \in A$ we need only find an open set U with $a \in U \subset A$ for which the theorem is true. Moreover we may assume that $g'(a) = I$.

Define $h\colon A \to \mathbf{R}^n$ by $h(x) = (g^1(x), \ldots, g^{n-1}(x), x^n)$. Then $h'(a) = I$. Hence in some open U' with $a \in U' \subset A$, the function h is 1-1 and $\det h'(x) \neq 0$. We can thus define $k\colon h(U') \to \mathbf{R}^n$ by $k(x) = (x^1, \ldots, x^{n-1}, g^n(h^{-1}(x)))$ and $g = k \circ h$. We have thus expressed g as the composition


x
h
$(g^1(x), \ldots, g^{n-1}(x), x^n)$
k
g(x)

of two maps, each of which changes fewer than n coordinates (Figure 3-3).

We must attend to a few details to ensure that k is a function of the proper sort. Since

$$(g^n \circ h^{-1})'(h(a)) = (g^n)'(a) \cdot [h'(a)]^{-1} = (g^n)'(a),$$

we have $D_n(g^n \circ h^{-1})(h(a)) = D_n g^n(a) = 1$, so that $k'(h(a)) = I$. Thus in some open set V with $h(a) \in V \subset h(U')$, the function k is 1-1 and $\det k'(x) \neq 0$. Letting $U = k^{-1}(V)$ we now have $g = k \circ h$, where $h\colon U \to \mathbf{R}^n$ and $k\colon V \to \mathbf{R}^n$ and $h(U) \subset V$. By (3) it suffices to prove the theorem for h and k. We give the proof for h; the proof for k is similar and easier.

Let $W \subset U$ be a rectangle of the form $D \times [a_n,b_n]$, where D is a rectangle in $\mathbf{R}^{n-1}$. By Fubini's theorem

$$\int_{h(W)} 1 = \int_{[a_n,b_n]} \left(\int_{h(D \times \{x^n\})} 1\, dx^1 \cdots dx^{n-1} \right) dx^n.$$

Let $h_{x^n}\colon D \to \mathbf{R}^{n-1}$ be defined by $h_{x^n}(x^1, \ldots, x^{n-1}) = (g^1(x^1, \ldots, x^n), \ldots, g^{n-1}(x^1, \ldots, x^n))$. Then each h_{x^n} is clearly 1-1 and

$$\det\,(h_{x^n})'(x^1, \ldots, x^{n-1}) = \det h'(x^1, \ldots, x^n) \neq 0.$$

Moreover

$$\int_{h(D \times \{x^n\})} 1\, dx^1 \cdots dx^{n-1} = \int_{h_{x^n}(D)} 1\, dx^1 \cdots dx^{n-1}.$$

Applying the theorem in the case $n - 1$ therefore gives

$$\begin{aligned}
\int_{h(W)} 1 &= \int_{[a_n,b_n]} \left(\int_{h_{x^n}(D)} 1\, dx^1 \cdots dx^{n-1} \right) dx^n \\
&= \int_{[a_n,b_n]} \left(\int_D |\det(h_{x^n})'(x^1, \ldots, x^{n-1})|\, dx^1 \cdots dx^{n-1} \right) dx^n \\
&= \int_{[a_n,b_n]} \left(\int_D |\det h'(x^1, \ldots, x^n)|\, dx^1 \cdots dx^{n-1} \right) dx^n \\
&= \int_W |\det h'|. \quad \blacksquare
\end{aligned}$$

The condition $\det g'(x) \neq 0$ may be eliminated from the

hypotheses of Theorem 3-13 by using the following theorem, which often plays an unexpected role.

3-14. Theorem (Sard's Theorem). *Let $g: A \to \mathbf{R}^n$ be continuously differentiable, where $A \subset \mathbf{R}^n$ is open, and let $B = \{x \in A: \det g'(x) = 0\}$. Then $g(B)$ has measure* 0.

Proof. Let $U \subset A$ be a closed rectangle such that all sides of U have length l, say. Let $\varepsilon > 0$. If N is sufficiently large and U is divided into N^n rectangles, with sides of length l/N, then for each of these rectangles S, if $x \in S$ we have

$$|Dg(x)(y - x) - g(y) - g(x)| < \varepsilon|x - y| \leq \varepsilon \sqrt{n}\,(l/N)$$

for all $y \in S$. If S intersects B we can choose $x \in S \cap B$; since $\det g'(x) = 0$, the set $\{Dg(x)(y - x): y \in S\}$ lies in an $(n - 1)$-dimensional subspace V of $\mathbf{R}^n$. Therefore the set $\{g(y) - g(x): y \in S\}$ lies within $\varepsilon \sqrt{n}\,(l/N)$ of V, so that $\{g(y): y \in S\}$ lies within $\varepsilon \sqrt{n}\,(l/N)$ of the $(n - 1)$-plane $V + g(x)$. On the other hand, by Lemma 2-10 there is a number M such that

$$|g(x) - g(y)| < M|x - y| \leq M \sqrt{n}\,(l/N).$$

Thus, if S intersects B, the set $\{g(y): y \in S\}$ is contained in a cylinder whose height is $< 2\varepsilon \sqrt{n}\,(l/N)$ and whose base is an $(n - 1)$-dimensional sphere of radius $< M \sqrt{n}\,(l/N)$. This cylinder has volume $< C(l/N)^n\varepsilon$ for some constant C. There are at most N^n such rectangles S, so $g(U \cap B)$ lies in a set of volume $< C(l/N)^n \cdot \varepsilon \cdot N^n = Cl^n \cdot \varepsilon$. Since this is true for all $\varepsilon > 0$, the set $g(U \cap B)$ has measure 0. Since (Problem 3-13) we can cover all of A with a sequence of such rectangles U, the desired result follows from Theorem 3-4. ▌

Theorem 3-14 is actually only the easy part of Sard's Theorem. The statement and proof of the deeper result will be found in [17], page 47.

Problems. **3-39.** Use Theorem 3-14 to prove Theorem 3-13 without the assumption $\det g'(x) \neq 0$.

3-40. If $g\colon \mathbf{R}^n \to \mathbf{R}^n$ and $\det g'(x) \neq 0$, prove that in some open set containing x we can write $g = T \circ g_n \circ \cdots \circ g_1$, where g_i is of the form $g_i(x) = (x^1, \ldots, f_i(x), \ldots, x^n)$, and T is a linear transformation. Show that we can write $g = g_n \circ \cdots \circ g_1$ if and only if $g'(x)$ is a diagonal matrix.

3-41. Define $f\colon \{r\colon r > 0\} \times (0,2\pi) \to \mathbf{R}^2$ by $f(r,\theta) = (r\cos\theta, r\sin\theta)$.

(a) Show that f is 1-1, compute $f'(r,\theta)$, and show that $\det f'(r,\theta) \neq 0$ for all (r,θ). Show that $f(\{r\colon r > 0\} \times (0,2\pi))$ is the set A of Problem 2-23.

(b) If $P = f^{-1}$, show that $P(x,y) = (r(x,y),\theta(x,y))$, where

$$r(x,y) = \sqrt{x^2 + y^2},$$
$$\theta(x,y) = \begin{cases} \arctan y/x & x > 0,\ y > 0, \\ \pi + \arctan y/x & x < 0, \\ 2\pi + \arctan y/x & x > 0,\ y < 0, \\ \pi/2 & x = 0,\ y > 0, \\ 3\pi/2 & x = 0,\ y < 0. \end{cases}$$

(Here arctan denotes the inverse of the function $\tan\colon (-\pi/2,\pi/2) \to \mathbf{R}$.) Find $P'(x,y)$. The function P is called the **polar coordinate system** on A.

(c) Let $C \subset A$ be the region between the circles of radii r_1 and r_2 and the half-lines through 0 which make angles of θ_1 and θ_2 with the x-axis. If $h\colon C \to \mathbf{R}$ is integrable and $h(x,y) = g(r(x,y),\theta(x,y))$, show that

$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} rg(r,\theta)\,d\theta\,dr.$$

If $B_r = \{(x,y)\colon x^2 + y^2 \leq r^2\}$, show that

$$\int_{B_r} h = \int_0^r \int_0^{2\pi} rg(r,\theta)\,d\theta\,dr.$$

(d) If $C_r = [-r,r] \times [-r,r]$, show that

$$\int_{B_r} e^{-(x^2+y^2)}\,dx\,dy = \pi(1 - e^{-r^2})$$

and

$$\int_{C_r} e^{-(x^2+y^2)}\,dx\,dy = \left(\int_{-r}^{r} e^{-x^2}\,dx\right)^2.$$

(e) Prove that

$$\lim_{r\to\infty} \int_{B_r} e^{-(x^2+y^2)}\,dx\,dy = \lim_{r\to\infty} \int_{C_r} e^{-(x^2+y^2)}\,dx\,dy$$

and conclude that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

"A mathematician is one to whom *that* is as obvious as that twice two makes four is to you. Liouville was a mathematician."

—LORD KELVIN

4

Integration on Chains

ALGEBRAIC PRELIMINARIES

If V is a vector space (over $\mathbf{R}$), we will denote the k-fold product $V \times \cdots \times V$ by V^k. A function $T: V^k \to \mathbf{R}$ is called **multilinear** if for each i with $1 \leq i \leq k$ we have

$$\begin{aligned} T(v_1, \ldots ,v_i + v_i', \ldots ,v_k) &= T(v_1, \ldots ,v_i, \ldots ,v_k) \\ &\quad + T(v_1, \ldots ,v_i', \ldots ,v_k), \\ T(v_1, \ldots ,av_i, \ldots ,v_k) &= aT(v_1, \ldots ,v_i, \ldots ,v_k). \end{aligned}$$

A multilinear function $T: V^k \to \mathbf{R}$ is called **a k-tensor** on V and the set of all k-tensors, denoted $\mathfrak{J}^k(V)$, becomes a vector space (over $\mathbf{R}$) if for $S,T \in \mathfrak{J}^k(V)$ and $a \in \mathbf{R}$ we define

$$\begin{aligned} (S + T)(v_1, \ldots ,v_k) &= S(v_1, \ldots ,v_k) + T(v_1, \ldots ,v_k), \\ (aS)(v_1, \ldots ,v_k) &= a \cdot S(v_1, \ldots ,v_k). \end{aligned}$$

There is also an operation connecting the various spaces $\mathfrak{J}^k(V)$. If $S \in \mathfrak{J}^k(V)$ and $T \in \mathfrak{J}^l(V)$, we define the **tensor product** $S \otimes T \in \mathfrak{J}^{k+l}(V)$ by

$$\begin{aligned} S \otimes T(v_1, \ldots ,v_k,v_{k+1}, \ldots ,v_{k+l}) \\ = S(v_1, \ldots ,v_k) \cdot T(v_{k+1}, \ldots ,v_{k+l}). \end{aligned}$$

Note that the order of the factors S and T is crucial here since $S \otimes T$ and $T \otimes S$ are far from equal. The following properties of $\otimes$ are left as easy exercises for the reader.

$$\begin{aligned}
(S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T,\\
S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2,\\
(aS) \otimes T &= S \otimes (aT) = a(S \otimes T),\\
(S \otimes T) \otimes U &= S \otimes (T \otimes U).
\end{aligned}$$

Both $(S \otimes T) \otimes U$ and $S \otimes (T \otimes U)$ are usually denoted simply $S \otimes T \otimes U$; higher-order products $T_1 \otimes \cdots \otimes T_r$ are defined similarly.

The reader has probably already noticed that $\mathfrak{J}^1(V)$ is just the dual space V^*. The operation $\otimes$ allows us to express the other vector spaces $\mathfrak{J}^k(V)$ in terms of $\mathfrak{J}^1(V)$.

4-1 Theorem. *Let $v_1, \ldots ,v_n$ be a basis for V, and let $\varphi_1, \ldots ,\varphi_n$ be the dual basis, $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products*

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \qquad 1 \le i_1, \ldots ,i_k \le n$$

is a basis for $\mathfrak{J}^k(V)$, which therefore has dimension n^k.

Proof. Note that

$$\begin{aligned}
\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, \ldots ,v_{j_k})
&= \delta_{i_1,j_1} \cdot \ldots \cdot \delta_{i_k,j_k}\\
&= \begin{cases} 1 & \text{if } j_1 = i_1, \ldots ,j_k = i_k,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

If $w_1, \ldots ,w_k$ are k vectors with $w_i = \Sigma_{j=1}^n a_{ij}v_j$ and T is in $\mathfrak{J}^k(V)$, then

$$\begin{aligned}
T(w_1, \ldots ,w_k) &= \sum_{j_1, \ldots ,j_k=1}^{n} a_{1,j_1} \cdot \ldots \cdot a_{k,j_k} T(v_{j_1}, \ldots ,v_{j_k})\\
&= \sum_{i_1, \ldots ,i_k=1}^{n} T(v_{i_1}, \ldots ,v_{i_k}) \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(w_1, \ldots ,w_k).
\end{aligned}$$

Thus $T = \Sigma_{i_1, \ldots ,i_k=1}^n T(v_{i_1}, \ldots ,v_{i_k}) \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$. Consequently the $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ span $\mathfrak{J}^k(V)$.

Suppose now that there are numbers $a_{i_1,\dots,i_k}$ such that

$$\sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0.$$

Applying both sides of this equation to $(v_{j_1}, \dots ,v_{j_k})$ yields $a_{j_1,\dots,j_k} = 0$. Thus the $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ are linearly independent. ▌

One important construction, familiar for the case of dual spaces, can also be made for tensors. If $f: V \to W$ is a linear transformation, a linear transformation $f^*: \mathfrak{J}^k(W) \to \mathfrak{J}^k(V)$ is defined by

$$f^*T(v_1, \dots ,v_k) = T(f(v_1), \dots ,f(v_k))$$

for $T \in \mathfrak{J}^k(W)$ and $v_1, \dots ,v_k \in V$. It is easy to verify that $f^*(S \otimes T) = f^*S \otimes f^*T$.

The reader is already familiar with certain tensors, aside from members of V^*. The first example is the inner product $\langle,\rangle \in \mathfrak{J}^2(\mathbf{R}^n)$. On the grounds that any good mathematical commodity is worth generalizing, we define an **inner product** on V to be a 2-tensor T such that T is **symmetric**, that is $T(v,w) = T(w,v)$ for $v,w \in V$ and such that T is **positive-definite**, that is, $T(v,v) > 0$ if $v \neq 0$. We distinguish $\langle,\rangle$ as the **usual inner product** on $\mathbf{R}^n$. The following theorem shows that our generalization is not too general.

4-2 Theorem. *If T is an inner product on V, there is a basis $v_1, \dots ,v_n$ for V such that $T(v_i,v_j) = \delta_{ij}$. (Such a basis is called* **orthonormal** *with respect to T.) Consequently there is an isomorphism $f: \mathbf{R}^n \to V$ such that $T(f(x),f(y)) = \langle x,y\rangle$ for $x,y \in \mathbf{R}^n$. In other words $f^*T = \langle,\rangle$.*

Proof. Let $w_1, \dots ,w_n$ be any basis for V. Define

$$w_1' = w_1,$$

$$w_2' = w_2 - \frac{T(w_1',w_2)}{T(w_1',w_1')} \cdot w_1',$$

$$w_3' = w_3 - \frac{T(w_1',w_3)}{T(w_1',w_1')} \cdot w_1' - \frac{T(w_2',w_3)}{T(w_2',w_2')} \cdot w_2',$$

etc.

It is easy to check that $T(w_i',w_j') = 0$ if $i \neq j$ and $w_i' \neq 0$ so that $T(w_i',w_i') > 0$. Now define $v_i = w_i'/\sqrt{T(w_i',w_i')}$. The isomorphism f may be defined by $f(e_i) = v_i$. ▮

Despite its importance, the inner product plays a far lesser role than another familiar, seemingly ubiquitous function, the tensor $\det \in \mathfrak{T}^n(\mathbf{R}^n)$. In attempting to generalize this function, we recall that interchanging two rows of a matrix changes the sign of its determinant. This suggests the following definition. A k-tensor $\omega \in \mathfrak{T}^k(V)$ is called **alternating** if

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) \qquad \text{for all } v_1, \ldots, v_k \in V.$$

(In this equation v_i and v_j are interchanged and all other v's are left fixed.) The set of all alternating k-tensors is clearly a subspace $\Lambda^k(V)$ of $\mathfrak{T}^k(V)$. Since it requires considerable work to produce the determinant, it is not surprising that alternating k-tensors are difficult to write down. There is, however, a uniform way of expressing all of them. Recall that the sign of a permutation σ, denoted $\operatorname{sgn} \sigma$, is $+1$ if σ is even and -1 if σ is odd. If $T \in \mathfrak{T}^k(V)$, we define $\operatorname{Alt}(T)$ by

$$\operatorname{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where S_k is the set of all permutations of the numbers 1 to k.

4-3 Theorem

(1) *If $T \in \mathfrak{T}^k(V)$, then $Alt(T) \in \Lambda^k(V)$.*
(2) *If $\omega \in \Lambda^k(V)$, then $Alt(\omega) = \omega$.*
(3) *If $T \in \mathfrak{T}^k(V)$, then $Alt(Alt(T)) = Alt(T)$.*

Proof

(1) Let (i,j) be the permutation that interchanges i and j and leaves all other numbers fixed. If $\sigma \in S_k$, let $\sigma' = \sigma \cdot (i,j)$. Then

$$
\begin{aligned}
&\mathrm{Alt}(T)(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)\\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(j)}, \ldots, v_{\sigma(i)}, \ldots, v_{\sigma(k)})\\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma'(1)}, \ldots, v_{\sigma'(i)}, \ldots, v_{\sigma'(j)}, \ldots, v_{\sigma'(k)})\\
&= \frac{1}{k!} \sum_{\sigma' \in S_k} -\operatorname{sgn} \sigma' \cdot T(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)})\\
&= -\mathrm{Alt}(T)(v_1, \ldots, v_k).
\end{aligned}
$$

(2) If $\omega \in \Lambda^k(V)$, and $\sigma = (i,j)$, then $\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn} \sigma \cdot \omega(v_1, \ldots, v_k)$. Since every σ is a product of permutations of the form (i,j), this equation holds of all σ. Therefore

$$
\begin{aligned}
\mathrm{Alt}(\omega)(v_1, \ldots, v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot \operatorname{sgn} \sigma \cdot \omega(v_1, \ldots, v_k)\\
&= \omega(v_1, \ldots, v_k).
\end{aligned}
$$

(3) follows immediately from (1) and (2). ▌

To determine the dimensions of $\Lambda^k(V)$, we would like a theorem analogous to Theorem 4-1. Of course, if $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$, then $\omega \otimes \eta$ is usually not in $\Lambda^{k+l}(V)$. We will therefore define a new product, the **wedge** product $\omega \wedge \eta \in \Lambda^{k+l}(V)$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!\, l!} \mathrm{Alt}(\omega \otimes \eta).$$

(The reason for the strange coefficient will appear later.) The following properties of $\wedge$ are left as an exercise for the reader:

$$
\begin{aligned}
(\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta,\\
\omega \wedge (\eta_1 + \eta_2) &= \omega \wedge \eta_1 + \omega \wedge \eta_2,\\
a\omega \wedge \eta = \omega \wedge a\eta &= a(\omega \wedge \eta),\\
\omega \wedge \eta &= (-1)^{kl} \eta \wedge \omega,\\
f^*(\omega \wedge \eta) &= f^*(\omega) \wedge f^*(\eta).
\end{aligned}
$$

The equation $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$ is true but requires more work.

4-4 Theorem

(1) *If* $S \in \mathfrak{J}^k(V)$ and $T \in \mathfrak{J}^l(V)$ *and* $Alt(S) = 0$, *then*

$$Alt(S \otimes T) = Alt(T \otimes S) = 0.$$

(2) $$Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta)).$$

(3) *If* $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, *and* $\theta \in \Lambda^m(V)$, *then*

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\,l!\,m!} Alt(\omega \otimes \eta \otimes \theta).$$

Proof

(1)

$$(k+l)!\, Alt(S \otimes T)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k+l}} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

If $G \subset S_{k+l}$ consists of all σ which leave $k+1, \ldots, k+l$ fixed, then

$$\sum_{\sigma \in G} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) = \left[\sum_{\sigma' \in S_k} \operatorname{sgn} \sigma' \cdot S(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)})\right] \cdot T(v_{k+1}, \ldots, v_{k+l}) = 0.$$

Suppose now that $\sigma_0 \notin G$. Let $G \cdot \sigma_0 = \{\sigma \cdot \sigma_0 : \sigma \in G\}$ and let $v_{\sigma_0(1)}, \ldots, v_{\sigma_0(k+l)} = w_1, \ldots, w_{k+l}$. Then

$$\sum_{\sigma \in G \cdot \sigma_0} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) = \left[\operatorname{sgn} \sigma_0 \cdot \sum_{\sigma' \in G} \operatorname{sgn} \sigma' \cdot S(w_{\sigma'(1)}, \ldots, w_{\sigma'(k)})\right] \cdot T(w_{k+1}, \ldots, w_{k+l}) = 0.$$

Notice that $G \cap G \cdot \sigma_0 = \varnothing$. In fact, if $\sigma \in G \cap G \cdot \sigma_0$, then $\sigma = \sigma' \cdot \sigma_0$ for some $\sigma' \in G$ and $\sigma_0 = \sigma \cdot (\sigma')^{-1} \in G$, a contradiction. We can then continue in this way, breaking S_{k+l} up into disjoint subsets; the sum over each subset is 0, so that the sum over S_{k+l} is 0. The relation $\text{Alt}(T \otimes S) = 0$ is proved similarly.

(2) We have

$$\text{Alt}(\text{Alt}(\eta \otimes \theta) - \eta \otimes \theta) = \text{Alt}(\eta \otimes \theta) - \text{Alt}(\eta \otimes \theta) = 0.$$

Hence by (1) we have

$$\begin{aligned} 0 &= \text{Alt}(\omega \otimes [\text{Alt}(\eta \otimes \theta) - \eta \otimes \theta]) \\ &= \text{Alt}(\omega \otimes \text{Alt}(\eta \otimes \theta)) - \text{Alt}(\omega \otimes \eta \otimes \theta). \end{aligned}$$

The other equality is proved similarly.

(3) $$\begin{aligned} (\omega \wedge \eta) \wedge \theta &= \frac{(k+l+m)!}{(k+l)!\,m!} \text{Alt}((\omega \wedge \eta) \otimes \theta) \\ &= \frac{(k+l+m)!}{(k+l)!\,m!} \frac{(k+l)!}{k!\,l!} \text{Alt}(\omega \otimes \eta \otimes \theta). \end{aligned}$$

The other equality is proved similarly. ∎

Naturally $\omega \wedge (\eta \wedge \theta)$ and $(\omega \wedge \eta) \wedge \theta$ are both denoted simply $\omega \wedge \eta \wedge \theta$, and higher-order products $\omega_1 \wedge \cdots \wedge \omega_r$ are defined similarly. If $v_1, \ldots, v_n$ is a basis for V and $\varphi_1, \ldots, \varphi_n$ is the dual basis, a basis for $\Lambda^k(V)$ can now be constructed quite easily.

4-5 Theorem. *The set of all*

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

is a basis for $\Lambda^k(V)$, which therefore has dimension

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. If $\omega \in \Lambda^k(V) \subset \mathfrak{J}^k(V)$, then we can write

$$\omega = \sum_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}.$$

Thus

$$\omega = \operatorname{Alt}(\omega) = \sum_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} \operatorname{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}).$$

Since each $\operatorname{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$ is a constant times one of the $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$, these elements span $\Lambda^k(V)$. Linear independence is proved as in Theorem 4-1 (cf. Problem 4-1). ▌

If V has dimension n, it follows from Theorem 4-5 that $\Lambda^n(V)$ has dimension 1. Thus all alternating n-tensors on V are multiples of any non-zero one. Since the determinant is an example of such a member of $\Lambda^n(\mathbf{R}^n)$, it is not surprising to find it in the following theorem.

4-6 Theorem. *Let $v_1, \ldots, v_n$ be a basis for V, and let $\omega \in \Lambda^n(V)$. If $w_i = \sum_{j=1}^n a_{ij} v_j$ are n vectors in V, then*

$$\omega(w_1, \ldots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \ldots, v_n).$$

Proof. Define $\eta \in \mathfrak{J}^n(\mathbf{R}^n)$ by

$$\eta((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn})) = \omega(\Sigma a_{1j} v_j, \ldots, \Sigma a_{nj} v_j).$$

Clearly $\eta \in \Lambda^n(\mathbf{R}^n)$ so $\eta = \lambda \cdot \det$ for some $\lambda \in \mathbf{R}$ and $\lambda = \eta(e_1, \ldots, e_n) = \omega(v_1, \ldots, v_n)$. ▌

Theorem 4-6 shows that a non-zero $\omega \in \Lambda^n(V)$ splits the bases of V into two disjoint groups, those with $\omega(v_1, \ldots, v_n) > 0$ and those for which $\omega(v_1, \ldots, v_n) < 0$; if $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases and $A = (a_{ij})$ is defined by $w_i = \Sigma a_{ij} v_j$, then $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are in the same group if and only if $\det A > 0$. This criterion is independent of ω and can always be used to divide the bases of V into two disjoint groups. Either of these two groups is called an **orientation** for V. The orientation to which a basis $v_1, \ldots, v_n$ belongs is denoted $[v_1, \ldots, v_n]$ and the

other orientation is denoted $-[v_1, \ldots, v_n]$. In $\mathbf{R}^n$ we define the **usual orientation** as $[e_1, \ldots, e_n]$.

The fact that $\dim \Lambda^n(\mathbf{R}^n) = 1$ is probably not new to you, since det is often defined as the unique element $\omega \in \Lambda^n(\mathbf{R}^n)$ such that $\omega(e_1, \ldots, e_n) = 1$. For a general vector space V there is no extra criterion of this sort to distinguish a particular $\omega \in \Lambda^n(V)$. Suppose, however, that an inner product T for V is given. If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases which are orthonormal with respect to T, and the matrix $A = (a_{ij})$ is defined by $w_i = \sum_{j=1}^n a_{ij}v_j$, then

$$\begin{aligned}\delta_{ij} = T(w_i,w_j) &= \sum_{k,l=1}^{n} a_{ik}a_{jl}T(v_k,v_l) \\ &= \sum_{k=1}^{n} a_{ik}a_{jk}.\end{aligned}$$

In other words, if $A^{\mathbf{T}}$ denotes the transpose of the matrix A, then we have $A \cdot A^{\mathbf{T}} = I$, so $\det A = \pm 1$. It follows from Theorem 4-6 that if $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \ldots, v_n) = \pm 1$, then $\omega(w_1, \ldots, w_n) = \pm 1$. If an orientation μ for V has also been given, it follows that there is a unique $\omega \in \Lambda^n(V)$ such that $\omega(v_1, \ldots, v_n) = 1$ whenever $v_1, \ldots, v_n$ is an orthonormal basis such that $[v_1, \ldots, v_n] = \mu$. This unique ω is called the **volume element** of V, determined by the inner product T and orientation μ. Note that det is the volume element of $\mathbf{R}^n$ determined by the usual inner product and usual orientation, and that $|\det(v_1, \ldots, v_n)|$ is the volume of the parallelipiped spanned by the line segments from 0 to each of $v_1, \ldots, v_n$.

We conclude this section with a construction which we will restrict to $\mathbf{R}^n$. If $v_1, \ldots, v_{n-1} \in \mathbf{R}^n$ and φ is defined by

$$\varphi(w) = \det \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ w \end{pmatrix},$$

then $\varphi \in \Lambda^1(\mathbf{R}^n)$; therefore there is a unique $z \in \mathbf{R}^n$ such that

$$\langle w,z \rangle = \varphi(w) = \det \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ w \end{pmatrix}$$

This z is denoted $v_1 \times \cdots \times v_{n-1}$ and called the **cross product** of $v_1, \ldots, v_{n-1}$. The following properties are immediate from the definition:

$$\begin{aligned} v_{\sigma(1)} \times \cdots \times v_{\sigma(n-1)} &= \operatorname{sgn} \sigma \cdot v_1 \times \cdots \times v_{n-1}, \\ v_1 \times \cdots \times av_i \times \cdots \times v_{n-1} &= a \cdot (v_1 \times \cdots \times v_{n-1}), \\ v_1 \times \cdots \times (v_i + v_i') \times \cdots \times v_{n-1} & \\ &= v_1 \times \cdots \times v_i \times \cdots \times v_{n-1} \\ &+ v_1 \times \cdots \times v_i' \times \cdots \times v_{n-1}. \end{aligned}$$

It is uncommon in mathematics to have a "product" that depends on more than two factors. In the case of two vectors $v,w \in \mathbf{R}^3$, we obtain a more conventional looking product, $v \times w \in \mathbf{R}^3$. For this reason it is sometimes maintained that the cross product can be defined only in $\mathbf{R}^3$.

Problems. **4-1.*** Let $e_1, \ldots, e_n$ be the usual basis of $\mathbf{R}^n$ and let $\varphi_1, \ldots, \varphi_n$ be the dual basis.

(a) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} (e_{i_1}, \ldots, e_{i_k}) = 1$. What would the right side be if the factor $(k + l)!/k!l!$ did not appear in the definition of $\wedge$?

(b) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \ldots, v_k)$ is the determinant of the $k \times k$ minor of $\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{pmatrix}$ obtained by selecting columns $i_1, \ldots, i_k$.

4-2. If $f\colon V \to V$ is a linear transformation and $\dim V = n$, then $f^*\colon \Lambda^n(V) \to \Lambda^n(V)$ must be multiplication by some constant c. Show that $c = \det f$.

4-3. If $\omega \in \Lambda^n(V)$ is the volume element determined by T and μ, and $w_1, \ldots, w_n \in V$, show that

$$|\omega(w_1, \ldots, w_n)| = \sqrt{\det(g_{ij})},$$

where $g_{ij} = T(w_i,w_j)$. *Hint:* If $v_1, \ldots, v_n$ is an orthonormal basis and $w_i = \sum_{j=1}^n a_{ij}v_j$, show that $g_{ij} = \sum_{k=1}^n a_{ik}a_{kj}$.

4-4. If ω is the volume element of V determined by T and μ, and $f\colon \mathbf{R}^n \to V$ is an isomorphism such that $f^*T = \langle,\rangle$ and such that $[f(e_1), \ldots, f(e_n)] = \mu$, show that $f^*\omega = \det$.

4-5. If $c\colon [0,1] \to (\mathbf{R}^n)^n$ is continuous and each $(c^1(t), \ldots, c^n(t))$ is a basis for $\mathbf{R}^n$, show that $[c^1(0), \ldots, c^n(0)] = [c^1(1), \ldots, c^n(1)]$. *Hint:* Consider $\det \circ c$.

4-6. (a) If $v \in \mathbf{R}^2$, what is $v \times$?

(b) If $v_1, \ldots, v_{n-1} \in \mathbf{R}^n$ are linearly independent, show that $[v_1, \ldots, v_{n-1}, v_1 \times \cdots \times v_{n-1}]$ is the usual orientation of $\mathbf{R}^n$.

4-7. Show that every non-zero $\omega \in \Lambda^n(V)$ is the volume element determined by some inner product T and orientation μ for V.

4-8. If $\omega \in \Lambda^n(V)$ is a volume element, define a "cross product" $v_1 \times \cdots \times v_{n-1}$ in terms of ω.

4-9.* Deduce the following properties of the cross product in $\mathbf{R}^3$:

(a)
$$\begin{array}{lll} e_1 \times e_1 = 0 & e_2 \times e_1 = -e_3 & e_3 \times e_1 = e_2 \\ e_1 \times e_2 = e_3 & e_2 \times e_2 = 0 & e_3 \times e_2 = -e_1 \\ e_1 \times e_3 = -e_2 & e_2 \times e_3 = e_1 & e_3 \times e_3 = 0. \end{array}$$

(b)
$$v \times w = (v^2w^3 - v^3w^2)e_1 + (v^3w^1 - v^1w^3)e_2 + (v^1w^2 - v^2w^1)e_3.$$

(c) $|v \times w| = |v| \cdot |w| \cdot |\sin\theta|$, where $\theta = \angle(v,w)$.
$\langle v \times w, v\rangle = \langle v \times w, w\rangle = 0$.

(d) $\langle v, w \times z\rangle = \langle w, z \times v\rangle = \langle z, v \times w\rangle$
$v \times (w \times z) = \langle v,z\rangle w - \langle v,w\rangle z$
$(v \times w) \times z = \langle v,z\rangle w - \langle w,z\rangle v$.

(e) $|v \times w| = \sqrt{\langle v,v\rangle \cdot \langle w,w\rangle - \langle v,w\rangle^2}$.

4-10. If $w_1, \ldots, w_{n-1} \in \mathbf{R}^n$, show that

$$|w_1 \times \cdots \times w_{n-1}| = \sqrt{\det(g_{ij})},$$

where $g_{ij} = \langle w_i,w_j\rangle$. *Hint:* Apply Problem 4-3 to a certain $(n-1)$-dimensional subspace of $\mathbf{R}^n$.

4-11. If T is an inner product on V, a linear transformation $f\colon V \to V$ is called **self-adjoint** (with respect to T) if $T(x,f(y)) = T(f(x),y)$ for $x,y \in V$. If $v_1, \ldots, v_n$ is an orthonormal basis and $A = (a_{ij})$ is the matrix of f with respect to this basis, show that $a_{ij} = a_{ji}$.

4-12. If $f_1, \ldots, f_{n-1}\colon \mathbf{R}^m \to \mathbf{R}^n$, define $f_1 \times \cdots \times f_{n-1}\colon \mathbf{R}^m \to \mathbf{R}^n$ by $f_1 \times \cdots \times f_{n-1}(p) = f_1(p) \times \cdots \times f_{n-1}(p)$. Use Problem 2-14 to derive a formula for $D(f_1 \times \cdots \times f_{n-1})$ when $f_1, \ldots, f_{n-1}$ are differentiable.

FIELDS AND FORMS

If $p \in \mathbf{R}^n$, the set of all pairs (p,v), for $v \in \mathbf{R}^n$, is denoted $\mathbf{R}^n{}_p$, and called the **tangent space** of $\mathbf{R}^n$ at p. This set is made into a vector space in the most obvious way, by defining

$$(p,v) + (p,w) = (p, v + w),$$
$$a \cdot (p,v) = (p,av).$$

A vector $v \in \mathbf{R}^n$ is often pictured as an arrow from 0 to v; the vector $(p,v) \in \mathbf{R}^n{}_p$ may be pictured (Figure 4-1) as an arrow with the same direction and length, but with initial point p. This arrow goes from p to the point $p + v$, and we therefore

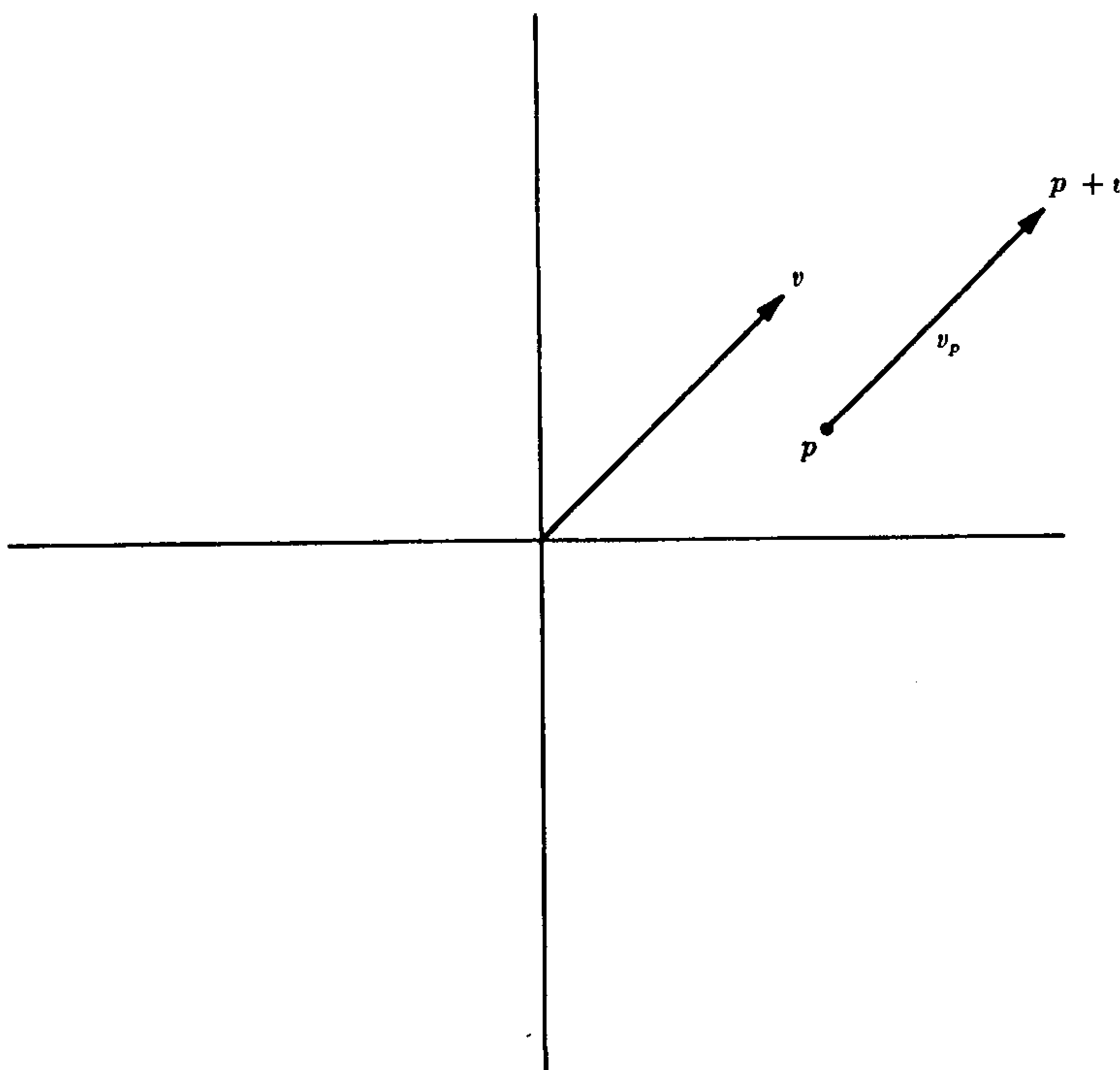


FIGURE 4-1

define $p + v$ to be the **end point** of (p,v). We will usually write (p,v) as v_p (read: the vector v at p).

The vector space $\mathbf{R}^n{}_p$ is so closely allied to $\mathbf{R}^n$ that many of the structures on $\mathbf{R}^n$ have analogues on $\mathbf{R}^n{}_p$. In particular the **usual inner product** $\langle,\rangle_p$ for $\mathbf{R}^n{}_p$ is defined by $\langle v_p,w_p\rangle_p = \langle v,w\rangle$, and the **usual orientation** for $\mathbf{R}^n{}_p$ is $[(e_1)_p, \ldots ,(e_n)_p]$.

Any operation which is possible in a vector space may be performed in each $\mathbf{R}^n{}_p$, and most of this section is merely an elaboration of this theme. About the simplest operation in a vector space is the selection of a vector from it. If such a selection is made in each $\mathbf{R}^n{}_p$, we obtain **a vector field** (Figure 4-2). To be precise, a vector field is a function F such that $F(p) \in \mathbf{R}^n{}_p$ for each $p \in \mathbf{R}^n$. For each p there are numbers $F^1(p), \ldots ,F^n(p)$ such that

$$F(p) = F^1(p) \cdot (e_1)_p + \cdots + F^n(p) \cdot (e_n)_p.$$

We thus obtain n **component functions** $F^i\colon \mathbf{R}^n \to \mathbf{R}$. The vector field F is called continuous, differentiable, etc., if the functions F^i are. Similar definitions can be made for a vector field defined only on an open subset of $\mathbf{R}^n$. Operations on vectors yield operations on vector fields when applied at each point separately. For example, if F and G are vector fields

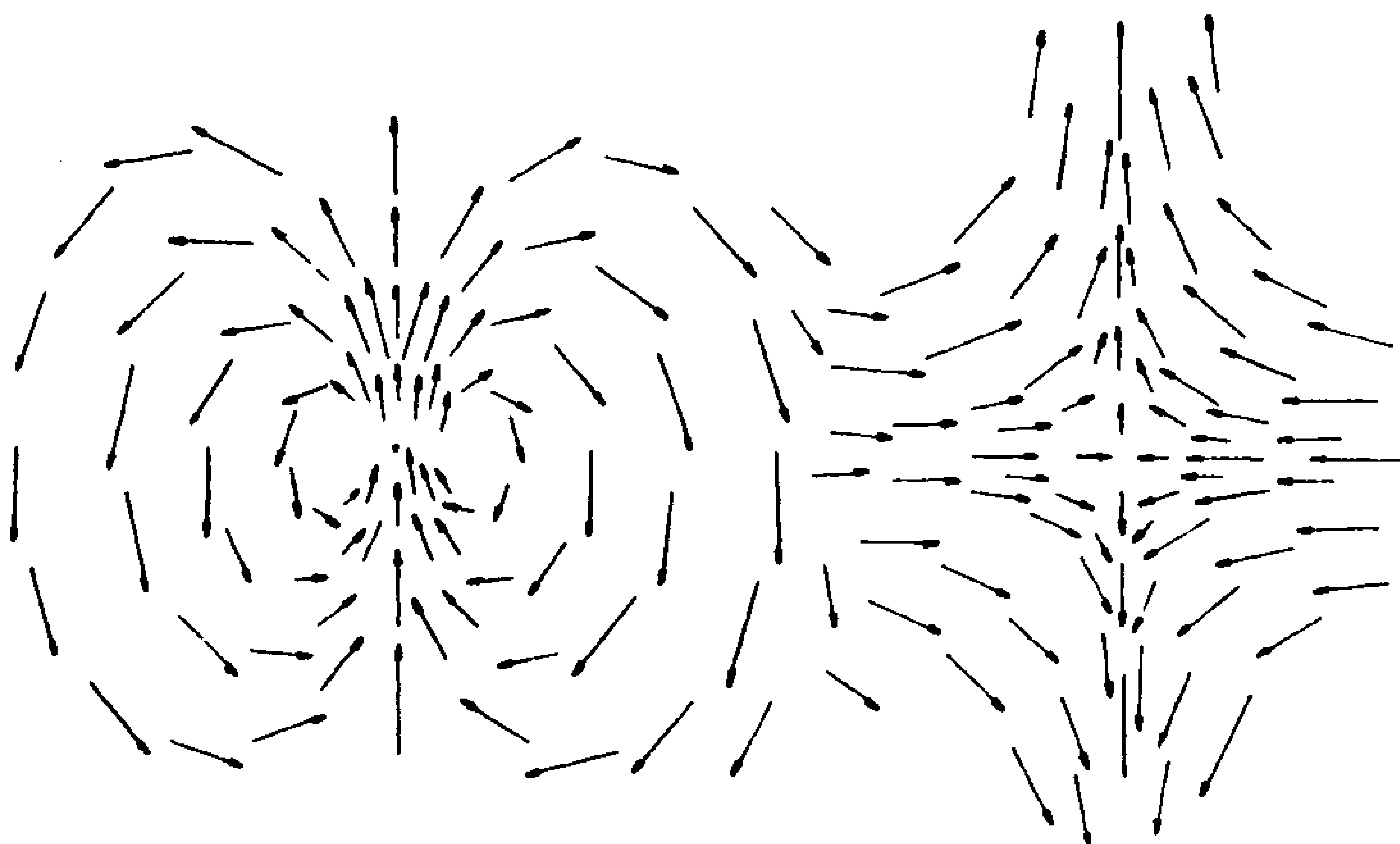


FIGURE 4-2

and f is a function, we define

$$(F + G)(p) = F(p) + G(p),$$
$$\langle F,G\rangle(p) = \langle F(p),G(p)\rangle,$$
$$(f \cdot F)(p) = f(p)F(p).$$

If $F_1, \ldots ,F_{n-1}$ are vector fields on $\mathbf{R}^n$, then we can similarly define

$$(F_1 \times \cdots \times F_{n-1})(p) = F_1(p) \times \cdots \times F_{n-1}(p).$$

Certain other definitions are standard and useful. We define the **divergence,** $\operatorname{div} F$ of F, as $\Sigma_{i=1}^n D_iF^i$. If we introduce the formal symbolism

$$\nabla = \sum_{i=1}^{n} D_i \cdot e_i,$$

we can write, symbolically, $\operatorname{div} F = \langle \nabla, F\rangle$. If $n = 3$ we write, in conformity with this symbolism,

$$\begin{aligned}(\nabla \times F)(p) = (D_2F^3 - D_3F^2)(e_1)_p \\ + (D_3F^1 - D_1F^3)(e_2)_p \\ + (D_1F^2 - D_2F^1)(e_3)_p.\end{aligned}$$

The vector field $\nabla \times F$ is called $\operatorname{curl} F$. The names "divergence" and "curl" are derived from physical considerations which are explained at the end of this book.

Many similar considerations may be applied to a function ω with $\omega(p) \in \Lambda^k(\mathbf{R}^n{}_p)$; such a function is called a ***k*-form** on $\mathbf{R}^n$, or simply a **differential form.** If $\varphi_1(p), \ldots ,\varphi_n(p)$ is the dual basis to $(e_1)_p, \ldots ,(e_n)_p$, then

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots ,i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)]$$

for certain functions $\omega_{i_1, \ldots ,i_k}$; the form ω is called continuous, differentiable, etc., if these functions are. We shall usually assume tacitly that forms and vector fields are differentiable, and "differentiable" will henceforth mean "C^∞"; this is a simplifying assumption that eliminates the need for counting how many times a function is differentiated in a proof. The sum $\omega + \eta$, product $f \cdot \omega$, and wedge product $\omega \wedge \eta$ are defined

in the obvious way. A function f is considered to be a 0-form and $f \cdot \omega$ is also written $f \wedge \omega$.

If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbf{R}^n)$. By a minor modification we therefore obtain a 1-form df, defined by

$$df(p)(v_p) = Df(p)(v).$$

Let us consider in particular the 1-forms $d\pi^i$. It is customary to let x^i denote the *function* π^i. (On $\mathbf{R}^3$ we often denote x^1, x^2, and x^3 by x, y, and z.) This standard notation has obvious disadvantages but it allows many classical results to be expressed by formulas of equally classical appearance. Since $dx^i(p)(v_p) = d\pi^i(p)(v_p) = D\pi^i(p)(v) = v^i$, we see that $dx^1(p), \ldots, dx^n(p)$ is just the dual basis to $(e_1)_p, \ldots, (e_n)_p$. Thus every k-form ω can be written

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The expression for df is of particular interest.

4-7 Theorem. *If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable, then*

$$df = D_1 f \cdot dx^1 + \cdots + D_n f \cdot dx^n.$$

In classical notation,

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n.$$

Proof.
$$\begin{aligned} df(p)(v_p) = Df(p)(v) &= \textstyle\sum_{i=1}^n v^i \cdot D_i f(p) \\ &= \textstyle\sum_{i=1}^n dx^i(p)(v_p) \cdot D_i f(p). \end{aligned}$$ ∎

If we consider now a differentiable function $f: \mathbf{R}^n \to \mathbf{R}^m$ we have a linear transformation $Df(p): \mathbf{R}^n \to \mathbf{R}^m$. Another minor modification therefore produces a linear transformation $f_*: \mathbf{R}^n{}_p \to \mathbf{R}^m{}_{f(p)}$ defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)}.$$

This linear transformation induces a linear transformation $f^*: \Lambda^k(\mathbf{R}^m{}_{f(p)}) \to \Lambda^k(\mathbf{R}^n{}_p)$. If ω is a k-form on $\mathbf{R}^m$ we can therefore define a k-form $f^*\omega$ on $\mathbf{R}^n$ by $(f^*\omega)(p) = f^*(\omega(f(p)))$.

Recall this means that if $v_1, \ldots, v_k \in \mathbf{R}^n{}_p$, then we have $f^*\omega(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k))$. As an antidote to the abstractness of these definitions we present a theorem, summarizing the important properties of f^*, which allows explicit calculations of $f^*\omega$.

4-8 Theorem. *If $f: \mathbf{R}^n \to \mathbf{R}^m$ is differentiable, then*

(1) $f^*(dx^i) = \Sigma_{j=1}^n D_jf^i \cdot dx^j = \Sigma_{j=1}^n \dfrac{\partial f^i}{\partial x^j}\, dx^j$.

(2) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$.

(3) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$.

(4) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Proof

(1)
$$\begin{aligned} f^*(dx^i)(p)(v_p) &= dx^i(f(p))(f_*v_p) \\ &= dx^i(f(p))(\Sigma_{j=1}^n v^j \cdot D_jf^1(p), \ldots, \Sigma_{j=1}^n v^j \cdot D_jf^m(p))_{f(p)} \\ &= \Sigma_{j=1}^n v^j \cdot D_jf^i(p) \\ &= \Sigma_{j=1}^n D_jf^i(p) \cdot dx^j(p)(v_p). \end{aligned}$$

The proofs of (2), (3), and (4) are left to the reader. ∎

By repeatedly applying Theorem 4-8 we have, for example,

$$\begin{aligned} f^*(P\, dx^1 \wedge dx^2 + Q\, dx^2 \wedge dx^3) = (P \circ f)[f^*(dx^1) \wedge f^*(dx^2)] \\ + (Q \circ f)[f^*(dx^2) \wedge f^*(dx^3)]. \end{aligned}$$

The expression obtained by expanding out each $f^*(dx^i)$ is quite complicated. (It is helpful to remember, however, that we have $dx^i \wedge dx^i = (-1)dx^i \wedge dx^i = 0$.) In one special case it will be worth our while to make an explicit evaluation.

4-9 Theorem. *If $f: \mathbf{R}^n \to \mathbf{R}^n$ is differentiable, then*

$$f^*(h\, dx^1 \wedge \cdots \wedge dx^n) = (h \circ f)(\det f')\, dx^1 \wedge \cdots \wedge dx^n.$$

Proof. Since

$$f^*(h\, dx^1 \wedge \cdots \wedge dx^n) = (h \circ f) f^*(dx^1 \wedge \cdots \wedge dx^n),$$

it suffices to show that

$$f^*(dx^1 \wedge \cdots \wedge dx^n) = (\det f')\, dx^1 \wedge \cdots \wedge dx^n.$$

Let $p \in \mathbf{R}^n$ and let $A = (a_{ij})$ be the matrix of $f'(p)$. Here, and whenever convenient and not confusing, we shall omit "p" in $dx^1 \wedge \cdots \wedge dx^n(p)$, etc. Then

$$\begin{aligned} f^*(dx^1 \wedge \cdots \wedge dx^n)(e_1, \ldots, e_n) &= dx^1 \wedge \cdots \wedge dx^n(f_*e_1, \ldots, f_*e_n) \\ &= dx^1 \wedge \cdots \wedge \quad \left(\sum_{i=1}^n a_{i1}e_i, \ldots, \sum_{i=1}^n a_{in}e_i\right) \\ &= \det(a_{ij}) \cdot dx^1 \wedge \cdots \wedge dx^n(e_1, \ldots, e_n), \end{aligned}$$

by Theorem 4-6. ▌

An important construction associated with forms is a generalization of the operator d which changes 0-forms into 1-forms. If

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

we define a $(k + 1)$-form $d\omega$, the **differential** of ω, by

$$\begin{aligned} d\omega &= \sum_{i_1 < \cdots < i_k} d\omega_{i_1, \ldots, i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^n D_\alpha(\omega_{i_1, \ldots, i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

4-10 Theorem

(1) $d(\omega + \eta) = d\omega + d\eta$.

(2) *If ω is a k-form and η is an l-form, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(3) $d(d\omega) = 0$. *Briefly,* $d^2 = 0$.

(4) *If ω is a k-form on $\mathbf{R}^m$ and $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is differentiable, then* $f^*(d\omega) = d(f^*\omega)$.

Proof

(1) Left to the reader.

(2) The formula is true if $\omega = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\eta = dx^{j_1} \wedge \cdots \wedge dx^{j_l}$, since all terms vanish. The formula is easily checked when ω is a 0-form. The general formula may be derived from (1) and these two observations.

(3) Since

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,\ldots,i_k}) dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

we have

$$d(d\omega) = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} D_{\alpha,\beta}(\omega_{i_1,\ldots,i_k}) dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

In this sum the terms

$$D_{\alpha,\beta}(\omega_{i_1,\ldots,i_k}) dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and

$$D_{\beta,\alpha}(\omega_{i_1,\ldots,i_k}) dx^\alpha \wedge dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

cancel in pairs.

(4) This is clear if ω is a 0-form. Suppose, inductively, that (4) is true when ω is a k-form. It suffices to prove (4) for a $(k+1)$-form of the type $\omega \wedge dx^i$. We have

$$\begin{aligned} f^*(d(\omega \wedge dx^i)) &= f^*(d\omega \wedge dx^i + (-1)^k \omega \wedge d(dx^i)) \\ &= f^*(d\omega \wedge dx^i) = f^*(d\omega) \wedge f^*(dx^i) \\ &= d(f^*\omega \wedge f^*(dx^i)) \qquad \text{by (2) and (3)} \\ &= d(f^*(\omega \wedge dx^i)). \ \blacksquare \end{aligned}$$

A form ω is called **closed** if $d\omega = 0$ and **exact** if $\omega = d\eta$, for some η. Theorem 4-10 shows that every exact form is closed, and it is natural to ask whether, conversely, every closed form is exact. If ω is the 1-form $P\,dx + Q\,dy$ on $\mathbf{R}^2$, then

$$\begin{aligned} d\omega &= (D_1P\,dx + D_2P\,dy) \wedge dx + (D_1Q\,dx + D_2Q\,dy) \wedge dy \\ &= (D_1Q - D_2P)dx \wedge dy. \end{aligned}$$

Thus, if $d\omega = 0$, then $D_1Q = D_2P$. Problems 2-21 and 3-34 show that there is a 0-form f such that $\omega = df = D_1f\,dx + D_2f\,dy$. If ω is defined only on a subset of $\mathbf{R}^2$, however, such a function may not exist. The classical example is the form

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

defined on $\mathbf{R}^2 - 0$. This form is usually denoted $d\theta$ (where θ is defined in Problem 3-41), since (Problem 4-21) it equals $d\theta$ on the set $\{(x,y)\colon x < 0$, or $x \geq 0$ and $y \neq 0\}$, where θ is defined. Note, however, that θ cannot be defined continuously on all of $\mathbf{R}^2 - 0$. If $\omega = df$ for some function $f\colon \mathbf{R}^2 - 0 \to \mathbf{R}$, then $D_1f = D_1\theta$ and $D_2f = D_2\theta$, so $f = \theta +$ constant, showing that such an f cannot exist.

Suppose that $\omega = \sum_{i=1}^n \omega_i\,dx^i$ is a 1-form on $\mathbf{R}^n$ and ω happens to equal $df = \sum_{i=1}^n D_if \cdot dx^i$. We can clearly assume that $f(0) = 0$. As in Problem 2-35, we have

$$\begin{aligned} f(x) &= \int_0^1 \frac{d}{dt} f(tx)\,dt \\ &= \int_0^1 \sum_{i=1}^n D_if(tx)\cdot x^i\,dt \\ &= \int_0^1 \sum_{i=1}^n \omega_i(tx)\cdot x^i\,dt. \end{aligned}$$

This suggests that in order to find f, given ω, we consider the function $I\omega$, defined by

$$I\omega(x) = \int_0^1 \sum_{i=1}^n \omega_i(tx)\cdot x^i\,dt.$$

Note that the definition of $I\omega$ makes sense if ω is defined only on an open set $A \subset \mathbf{R}^n$ with the property that whenever $x \in A$, the line segment from 0 to x is contained in A; such an open set is called **star-shaped** with respect to 0 (Figure 4-3). A somewhat involved calculation shows that (on a star-shaped open set) we have $\omega = d(I\omega)$ provided that ω satisfies the necessary condition $d\omega = 0$. The calculation, as well as the definition of $I\omega$, may be generalized considerably:

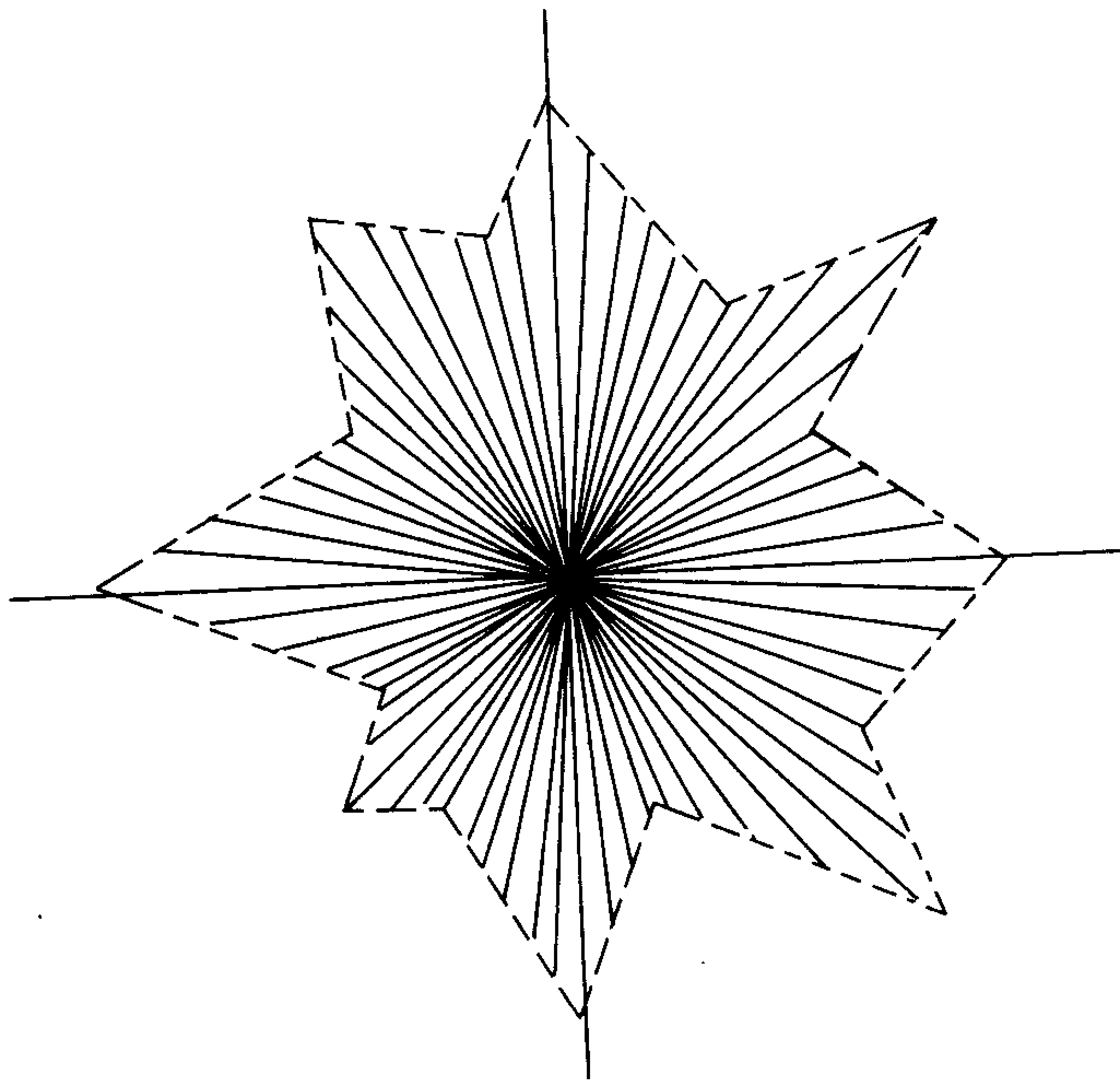


FIGURE 4-3

4-11 Theorem (Poincaré Lemma). *If $A \subset \mathbf{R}^n$ is an open set star-shaped with respect to 0, then every closed form on A is exact.*

Proof. We will define a function I from l-forms to $(l-1)$-forms (for each l), such that $I(0) = 0$ and $\omega = I(d\omega) + d(I\omega)$ for any form ω. It follows that $\omega = d(I\omega)$ if $d\omega = 0$. Let

$$\omega = \sum_{i_1 < \cdots < i_l} \omega_{i_1, \ldots, i_l}\, dx^{i_1} \wedge \cdots \wedge dx^{i_l}.$$

Since A is star-shaped we can define

$$I\omega(x) = \sum_{i_1 < \cdots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l-1} \omega_{i_1, \ldots, i_l}(tx)\, dt \right) x^{i_\alpha}$$
$$dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}.$$

(The symbol $\frown$ over dx^{i_α} indicates that it is omitted.) The

proof that $\omega = I(d\omega) + d(I\omega)$ is an elaborate computation: We have, using Problem 3-32,

$$d(I\omega) = l \cdot \sum_{i_1 < \cdots < i_l} \left(\int_0^1 t^{l-1} \omega_{i_1, \ldots, i_l}(tx)dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_l}$$
$$+ \sum_{i_1 < \cdots < i_l} \sum_{\alpha=1}^{l} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^l D_j(\omega_{i_1, \ldots, i_l})(tx)dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}.$$

(Explain why we have the factor t^l, instead of t^{l-1}.) We also have

$$d\omega = \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{n} D_j(\omega_{i_1, \ldots, i_l}) \cdot dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_l}.$$

Applying I to the $(l + 1)$-form $d\omega$, we obtain

$$I(d\omega) = \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l D_j(\omega_{i_1, \ldots, i_l})(tx)dt \right) x^j dx^{i_1} \wedge \cdots \wedge dx^{i_l}$$
$$- \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^l D_j(\omega_{i_1, \ldots, i_l})(tx)dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}.$$

Adding, the triple sums cancel, and we obtain

$$\begin{aligned} d(I\omega) + I(d\omega) &= \sum_{i_1 < \cdots < i_l} l \cdot \left(\int_0^1 t^{l-1} \omega_{i_1, \ldots, i_l}(tx)dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_l} \\ &\quad + \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l x^j D_j(\omega_{i_1, \ldots, i_l})(tx)dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_l} \\ &= \sum_{i_1 < \cdots < i_l} \left(\int_0^1 \frac{d}{dt} [t^l \omega_{i_1, \ldots, i_l}(tx)]dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_l} \\ &= \sum_{i_1 < \cdots < i_l} \omega_{i_1, \ldots, i_l} dx^{i_1} \wedge \cdots \wedge dx^{i_l} \\ &= \omega. \quad \blacksquare \end{aligned}$$

Problems. **4-13.** (a) If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ and $g\colon \mathbf{R}^m \to \mathbf{R}^p$, show that $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$.

(b) If $f,g\colon \mathbf{R}^n \to \mathbf{R}$, show that $d(f \cdot g) = f \cdot dg + g \cdot df$.

4-14. Let c be a differentiable curve in $\mathbf{R}^n$, that is, a differentiable function $c\colon [0,1] \to \mathbf{R}^n$. Define the **tangent vector** v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \ldots, (c^n)'(t))_{c(t)}$. If $f\colon \mathbf{R}^n \to \mathbf{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.

4-15. Let $f\colon \mathbf{R} \to \mathbf{R}$ and define $c\colon \mathbf{R} \to \mathbf{R}^2$ by $c(t) = (t,f(t))$. Show that the end point of the tangent vector of c at t lies on the tangent line to the graph of f at $(t,f(t))$.

4-16. Let $c\colon [0,1] \to \mathbf{R}^n$ be a curve such that $|c(t)| = 1$ for all t. Show that $c(t)_{c(t)}$ and the tangent vector to c at t are perpendicular.

4-17. If $f\colon \mathbf{R}^n \to \mathbf{R}^n$, define a vector field $\mathbf{f}$ by $\mathbf{f}(p) = f(p)_p \in \mathbf{R}^n{}_p$.

(a) Show that every vector field F on $\mathbf{R}^n$ is of the form $\mathbf{f}$ for some f.

(b) Show that $\operatorname{div} \mathbf{f} = \operatorname{trace} f'$.

4-18. If $f\colon \mathbf{R}^n \to \mathbf{R}$, define a vector field $\operatorname{grad} f$ by

$$(\operatorname{grad} f)(p) = D_1 f(p) \cdot (e_1)_p + \cdots + D_n f(p) \cdot (e_n)_p.$$

For obvious reasons we also write $\operatorname{grad} f = \nabla f$. If $\nabla f(p) = w_p$, prove that $D_v f(p) = \langle v,w \rangle$ and conclude that $\nabla f(p)$ is the direction in which f is changing fastest at p.

4-19. If F is a vector field on $\mathbf{R}^3$, define the forms

$$\omega_F^1 = F^1\,dx + F^2\,dy + F^3\,dz,$$
$$\omega_F^2 = F^1\,dy \wedge dz + F^2\,dz \wedge dx + F^3\,dx \wedge dy.$$

(a) Prove that

$$df = \omega^1_{\operatorname{grad} f},$$
$$d(\omega_F^1) = \omega^2_{\operatorname{curl} F},$$
$$d(\omega_F^2) = (\operatorname{div} F)\,dx \wedge dy \wedge dz.$$

(b) Use (a) to prove that

$$\operatorname{curl} \operatorname{grad} f = 0,$$
$$\operatorname{div} \operatorname{curl} F = 0.$$

(c) If F is a vector field on a star-shaped open set A and $\operatorname{curl} F = 0$, show that $F = \operatorname{grad} f$ for some function $f\colon A \to \mathbf{R}$. Similarly, if $\operatorname{div} F = 0$, show that $F = \operatorname{curl} G$ for some vector field G on A.

4-20. Let $f\colon U \to \mathbf{R}^n$ be a differentiable function with a differentiable inverse $f^{-1}\colon f(U) \to \mathbf{R}^n$. If every closed form on U is exact, show that the same is true for $f(U)$. *Hint:* If $d\omega = 0$ and $f^*\omega = d\eta$, consider $(f^{-1})^*\eta$.

4-21.* Prove that on the set where θ is defined we have

$$d\theta = \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy.$$

GEOMETRIC PRELIMINARIES

A **singular n-cube** in $A \subset \mathbf{R}^n$ is a continuous function c: $[0,1]^n \to A$ (here $[0,1]^n$ denotes the n-fold product $[0,1] \times \cdots \times [0,1]$). We let $\mathbf{R}^0$ and $[0,1]^0$ both denote $\{0\}$. A singular 0-cube in A is then a function f: $\{0\} \to A$ or, what amounts to the same thing, a point in A. A singular 1-cube is often called a **curve.** A particularly simple, but particularly important example of a singular n-cube in $\mathbf{R}^n$ is the **standard n-cube** I^n: $[0,1]^n \to \mathbf{R}^n$ defined by $I^n(x) = x$ for $x \in [0,1]^n$.

We shall need to consider formal sums of singular n-cubes in A multiplied by integers, that is, expressions like

$$2c_1 + 3c_2 - 4c_3,$$

where c_1, c_2, c_3 are singular n-cubes in A. Such a finite sum of singular n-cubes with integer coefficients is called an **n-chain** in A. In particular a singular n-cube c is also considered as an n-chain $1 \cdot c$. It is clear how n-chains can be added, and multiplied by integers. For example

$$2(c_1 + 3c_4) + (-2)(c_1 + c_3 + c_2) = -2c_2 - 2c_3 + 6c_4.$$

(A rigorous exposition of this formalism is presented in Problem 4-22.)

For each singular n-chain c in A we shall define an $(n-1)$-chain in A called the **boundary** of c and denoted ∂c. The boundary of I^2, for example, might be defined as the sum of four singular 1-cubes arranged counterclockwise around the boundary of $[0,1]^2$, as indicated in Figure 4-4(a). It is actually much more convenient to define ∂I^2 as the sum, with the indicated coefficients, of the four singular 1-cubes shown in Figure 4-4(b). The precise definition of ∂I^n requires some preliminary notions. For each i with $1 \le i \le n$ we define two singular $(n-1)$-cubes $I^n_{(i,0)}$ and $I^n_{(i,1)}$ as follows. If

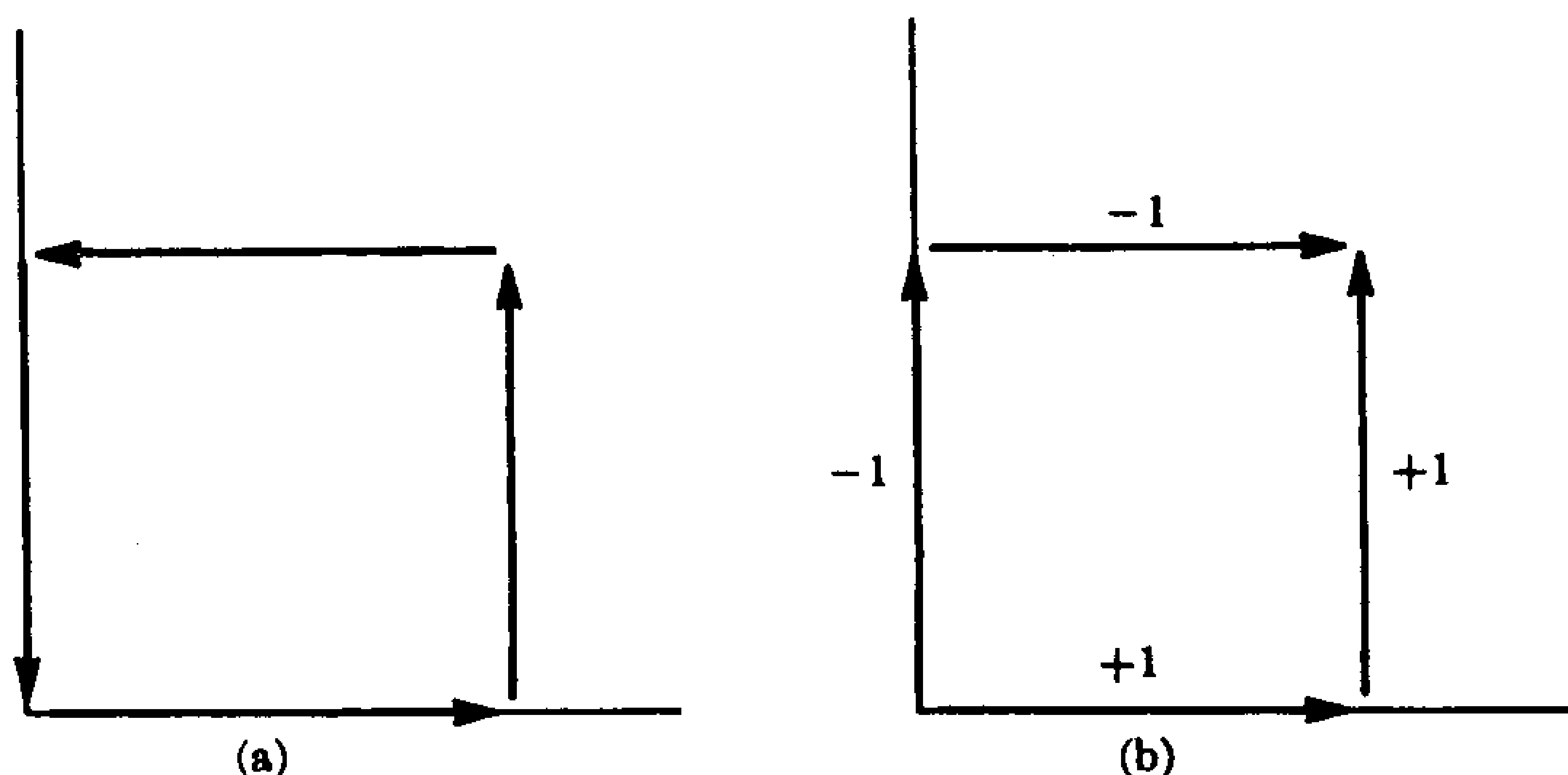


FIGURE 4-4

$x \in [0,1]^{n-1}$, then

$$\begin{aligned} I^n_{(i,0)}(x) &= I^n(x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{n-1}) \\ &= (x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{n-1}), \\ I^n_{(i,1)}(x) &= I^n(x^1, \ldots, x^{i-1}, 1, x^i, \ldots, x^{n-1}) \\ &= (x^1, \ldots, x^{i-1}, 1, x^i, \ldots, x^{n-1}). \end{aligned}$$

We call $I^n_{(i,0)}$ the $(i,0)$-face of I^n and $I^n_{(i,1)}$ the $(i,1)$-face (Figure 4-5). We then define

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.$$

For a general singular n-cube $c\colon [0,1]^n \to A$ we first define the (i,α)-face,

$$c_{(i,\alpha)} = c \circ (I^n_{(i,\alpha)})$$

and then define

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}.$$

Finally we define the boundary of an n-chain $\Sigma a_i c_i$ by

$$\partial(\Sigma a_i c_i) = \Sigma a_i \partial(c_i).$$

Although these few definitions suffice for all applications in this book, we include here the one standard property of ∂.

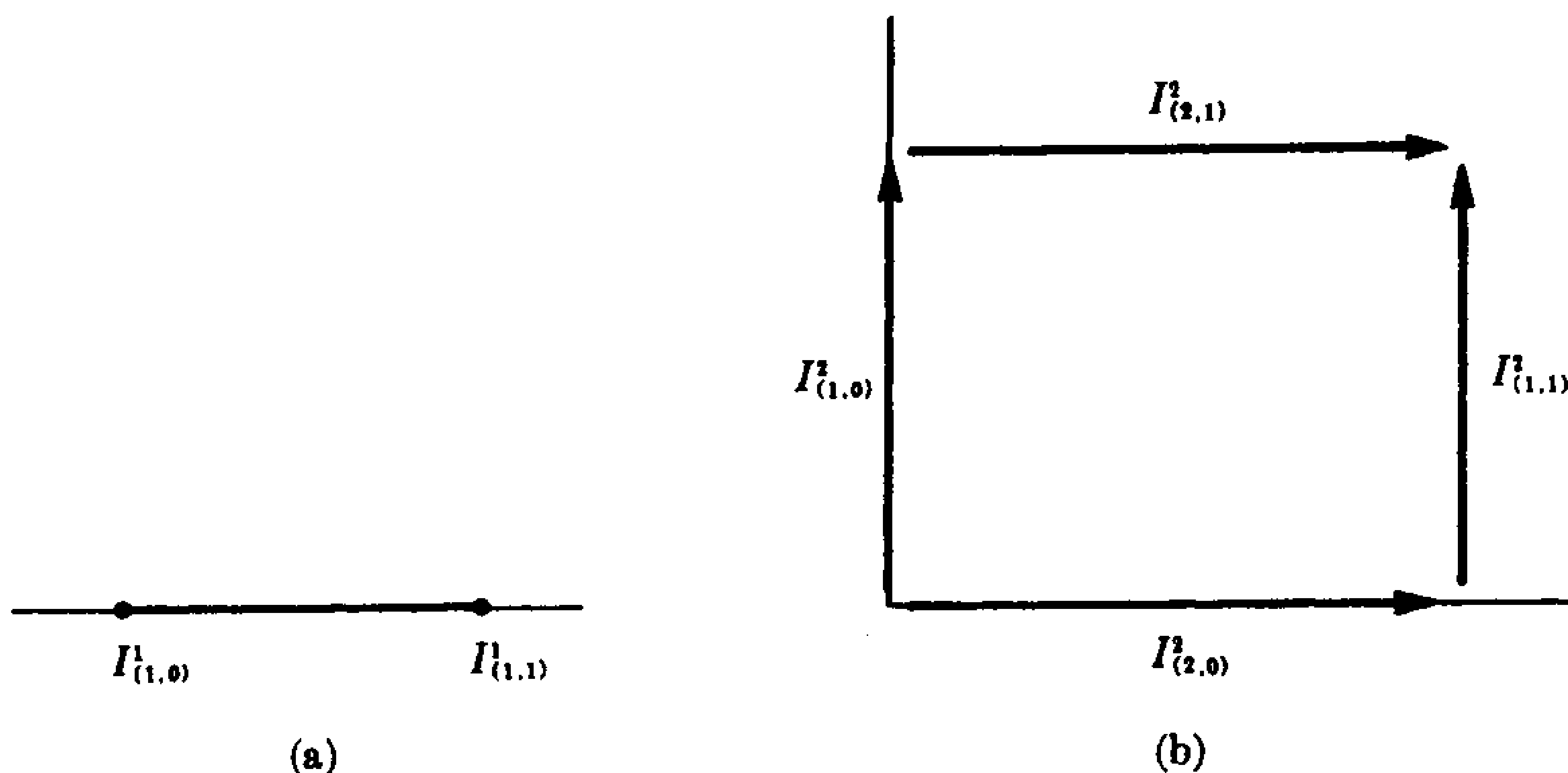


FIGURE 4-5

4-12 Theorem. *If c is an n-chain in A, then $\partial(\partial c) = 0$. Briefly, $\partial^2 = 0$.*

Proof. Let $i \le j$ and consider $(I^n_{(i,\alpha)})_{(j,\beta)}$. If $x \in [0,1]^{n-2}$, then, remembering the definition of the (j,β)-face of a singular n-cube, we have

$$\begin{aligned}(I^n_{(i,\alpha)})_{(j,\beta)}(x) &= I^n_{(i,\alpha)}(I^{n-1}_{(j,\beta)}(x)) \\ &= I^n_{(i,\alpha)}(x^1, \ldots, x^{j-1}, \beta, x^j, \ldots, x^{n-2}) \\ &= I^n(x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^{j-1}, \beta, x^j, \ldots, x^{n-2}).\end{aligned}$$

Similarly

$$\begin{aligned}(I^n_{(j+1,\beta)})_{(i,\alpha)} &= I^n_{(j+1,\beta)}(I^{n-1}_{(i,\alpha)}(x)) \\ &= I^n_{(j+1,\beta)}(x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^{n-2}) \\ &= I^n(x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^{j-1}, \beta, x^j, \ldots, x^{n-2}).\end{aligned}$$

Thus $(I^n_{(i,\alpha)})_{(j,\beta)} = (I^n_{(j+1,\beta)})_{(i,\alpha)}$ for $i \le j$. (It may help to verify this in Figure 4-5.) It follows easily for any singular n-cube c that $(c_{(i,\alpha)})_{(j,\beta)} = (c_{(j+1,\beta)})_{(i,\alpha)}$ when $i \le j$. Now

$$\begin{aligned}\partial(\partial c) &= \partial\left(\sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}\right) \\ &= \sum_{i=1}^{n} \sum_{\alpha=0,1} \sum_{j=1}^{n-1} \sum_{\beta=0,1} (-1)^{i+\alpha+j+\beta} (c_{(i,\alpha)})_{(j,\beta)}.\end{aligned}$$

In this sum $(c_{(i,\alpha)})_{(j,\beta)}$ and $(c_{(j+1,\beta)})_{(i,\alpha)}$ occur with opposite signs. Therefore all terms cancel out in pairs and $\partial(\partial c) = 0$. Since the theorem is true for any singular n-cube, it is also true for singular n-chains. ∎

It is natural to ask whether Theorem 4-12 has a converse: If $\partial c = 0$, is there a chain d in A such that $c = \partial d$? The answer depends on A and is generally "no." For example, define $c\colon [0,1] \to \mathbf{R}^2 - 0$ by $c(t) = (\sin 2\pi nt, \cos 2\pi nt)$, where n is a non-zero integer. Then $c(1) = c(0)$, so $\partial c = 0$. But (Problem 4-26) there is no 2-chain c' in $\mathbf{R}^2 - 0$, with $\partial c' = c$.

Problems. **4-22.** Let $\mathcal{S}$ be the set of all singular n-cubes, and $\mathbf{Z}$ the integers. An ***n*-chain** is a function $f\colon \mathcal{S} \to \mathbf{Z}$ such that $f(c) = 0$ for all but finitely many c. Define $f + g$ and nf by $(f + g)(c) = f(c) + g(c)$ and $nf(c) = n \cdot f(c)$. Show that $f + g$ and nf are n-chains if f and g are. If $c \in \mathcal{S}$, let c also denote the function f such that $f(c) = 1$ and $f(c') = 0$ for $c' \neq c$. Show that every n-chain f can be written $a_1c_1 + \cdots + a_kc_k$ for some integers $a_1, \ldots, a_k$ and singular n-cubes $c_1, \ldots, c_k$.

4-23. For $R > 0$ and n an integer, define the singular 1-cube $c_{R,n}\colon [0,1] \to \mathbf{R}^2 - 0$ by $c_{R,n}(t) = (R \cos 2\pi nt, R \sin 2\pi nt)$. Show that there is a singular 2-cube $c\colon [0,1]^2 \to \mathbf{R}^2 - 0$ such that $c_{R_1,n} - c_{R_2,n} = \partial c$.

4-24. If c is a singular 1-cube in $\mathbf{R}^2 - 0$ with $c(0) = c(1)$, show that there is an integer n such that $c - c_{1,n} = \partial c^2$ for some 2-chain c^2. *Hint:* First partition $[0,1]$ so that each $c([t_{i-1},t_i])$ is contained on one side of some line through 0.

THE FUNDAMENTAL THEOREM OF CALCULUS

The fact that $d^2 = 0$ and $\partial^2 = 0$, not to mention the typographical similarity of d and ∂, suggests some connection between chains and forms. This connection is established by integrating forms over chains. Henceforth only differentiable singular n-cubes will be considered.

If ω is a k-form on $[0,1]^k$, then $\omega = f\,dx^1 \wedge \cdots \wedge dx^k$ for a unique function f. We define

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f.$$

We could also write this as

$$\int_{[0,1]^k} f\,dx^1 \wedge \cdots \wedge dx^k = \int_{[0,1]^k} f(x^1, \ldots ,x^k)dx^1 \cdots dx^k,$$

one of the reasons for introducing the functions x^i.

If ω is a k-form on A and c is a singular k-cube in A, we define

$$\int_c \omega = \int_{[0,1]^k} c^*\omega.$$

Note, in particular, that

$$\begin{aligned}\int_{I^k} f\,dx^1 \wedge \cdots \wedge dx^k &= \int_{[0,1]^k} (I^k)^*(f\,dx^1 \wedge \cdots \wedge dx^k)\\ &= \int_{[0,1]^k} f(x^1, \ldots ,x^k)dx^1 \cdots dx^k.\end{aligned}$$

A special definition must be made for $k = 0$. A 0-form ω is a function; if $c\colon \{0\} \to A$ is a singular 0-cube in A we define

$$\int_c \omega = \omega(c(0)).$$

The integral of ω over a k-chain $c = \Sigma a_i c_i$ is defined by

$$\int_c \omega = \sum a_i \int_{c_i} \omega.$$

The integral of a 1-form over a 1-chain is often called a **line integral.** If $P\,dx + Q\,dy$ is a 1-form on $\mathbf{R}^2$ and $c\colon [0,1] \to \mathbf{R}^2$ is a singular 1-cube (a curve), then one can (but we will not) prove that

$$\begin{aligned}\int_c P\,dx + Q\,dy = \lim \sum_{i=1}^{n} &[c^1(t_i) - c^1(t_{i-1})] \cdot P(c(t^i))\\ &+ [c^2(t_i) - c^2(t_{i-1})] \cdot Q(c(t^i))\end{aligned}$$

where $t_0, \ldots ,t_n$ is a partition of $[0,1]$, the choice of t^i in $[t_{i-1},t_i]$ is arbitrary, and the limit is taken over all partitions

as the maximum of $|t_i - t_{i-1}|$ goes to 0. The right side is often taken as a definition of $\int_c P\,dx + Q\,dy$. This is a natural definition to make, since these sums are very much like the sums appearing in the definition of ordinary integrals. However such an expression is almost impossible to work with and is quickly equated with an integral equivalent to $\int_{[0,1]} c^*(P\,dx + Q\,dy)$. Analogous definitions for **surface integrals,** that is, integrals of 2-forms over singular 2-cubes, are even more complicated and difficult to use. This is one reason why we have avoided such an approach. The other reason is that the definition given here is the one that makes sense in the more general situations considered in Chapter 5.

The relationship between forms, chains, d, and ∂ is summed up in the neatest possible way by Stokes' theorem, sometimes called the fundamental theorem of calculus in higher dimensions (if $k = 1$ and $c = I^1$, it really is the fundamental theorem of calculus).

4-13 Theorem (Stokes' Theorem). *If ω is a $(k-1)$-form on an open set $A \subset \mathbf{R}^n$ and c is a k-chain in A, then*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof. Suppose first that $c = I^k$ and ω is a $(k-1)$-form on $[0,1]^k$. Then ω is the sum of $(k-1)$-forms of the type

$$f\,dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k,$$

and it suffices to prove the theorem for each of these. This simply involves a computation:

Note that

$$\int_{[0,1]^{k-1}} I_{(j,\alpha)}^k{}^*(f\,dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k) = \begin{cases} 0 & \text{if } j \neq i, \\ \displaystyle\int_{[0,1]^k} f(x^1, \ldots, \alpha, \ldots, x^k)dx^1 \cdots dx^k & \text{if } j = i. \end{cases}$$

Therefore

$$\int_{\partial I^k} f\,dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} I^{k}_{(j,\alpha)}{}^*(f\,dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

$$= (-1)^{i+1} \int_{[0,1]^k} f(x^1, \ldots, 1, \ldots, x^k)dx^1 \cdots dx^k + (-1)^i \int_{[0,1]^k} f(x^1, \ldots, 0, \ldots, x^k)dx^1 \cdots dx^k.$$

On the other hand,

$$\int_{I^k} d(f\,dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

$$= \int_{[0,1]^k} D_i f\,dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k$$

$$= (-1)^{i-1} \int_{[0,1]^k} D_i f.$$

By Fubini's theorem and the fundamental theorem of calculus (in one dimension) we have

$$\int_{I^k} d(f\,dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

$$= (-1)^{i-1} \int_0^1 \cdots \left(\int_0^1 D_i f(x^1, \ldots, x^k)dx^i\right) dx^1 \cdots \widehat{dx^i} \cdots dx^k$$

$$= (-1)^{i-1} \int_0^1 \cdots \int_0^1 [f(x^1, \ldots, 1, \ldots, x^k) - f(x^1, \ldots, 0, \ldots, x^k)]dx^1 \cdots \widehat{dx^i} \cdots dx^k$$

$$= (-1)^{i-1} \int_{[0,1]^k} f(x^1, \ldots, 1, \ldots, x^k)dx^1 \cdots dx^k + (-1)^i \int_{[0,1]^k} f(x^1, \ldots, 0, \ldots, x^k)dx^1 \cdots dx^k.$$

Thus

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega.$$

If c is an arbitrary singular k-cube, working through the definitions will show that

$$\int_{\partial c} \omega = \int_{\partial I^k} c^*\omega.$$

Therefore

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega.$$

Finally, if c is a k-chain $\Sigma a_i c_i$, we have

$$\int_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int_{\partial c_i} \omega = \int_{\partial c} \omega. \blacksquare$$

Stokes' theorem shares three important attributes with many fully evolved major theorems:

1. It is trivial.
2. It is trivial because the terms appearing in it have been properly defined.
3. It has significant consequences.

Since this entire chapter was little more than a series of definitions which made the statement and proof of Stokes' theorem possible, the reader should be willing to grant the first two of these attributes to Stokes' theorem. The rest of the book is devoted to justifying the third.

Problems. **4-25.** (*Independence of parameterization*). Let c be a singular k-cube and p: $[0,1]^k \to [0,1]^k$ a 1-1 function such that $p([0,1]^k) = [0,1]^k$ and $\det p'(x) \geq 0$ for $x \in [0,1]^k$. If ω is a k-form, show that

$$\int_c \omega = \int_{c \circ p} \omega.$$

4-26. Show that $\int_{c_{R,n}} d\theta = 2\pi n$, and use Stokes' theorem to conclude that $c_{R,n} \neq \partial c$ for any 2-chain c in $\mathbf{R}^2 - 0$ (recall the definition of $c_{R,n}$ in Problem 4-23).

4-27. Show that the integer n of Problem 4-24 is unique. This integer is called the **winding number** of c around 0.

4-28. Recall that the set of complex numbers $\mathbf{C}$ is simply $\mathbf{R}^2$ with $(a,b) = a + bi$. If $a_1, \ldots, a_n \in \mathbf{C}$ let f: $\mathbf{C} \to \mathbf{C}$ be $f(z) = z^n + a_1z^{n-1} + \cdots + a_n$. Define the singular 1-cube $c_{R,f}$:

$[0,1] \to \mathbf{C} - 0$ by $c_{R,f} = f \circ c_{R,1}$, and the singular 2-cube c by $c(s,t) = t \cdot c_{R,n}(s) + (1-t)c_{R,f}(s)$.

(a) Show that $\partial c = c_{R,f} - c_{R,n}$, and that $c([0,1] \times [0,1]) \subset \mathbf{C} - 0$ if R is large enough.

(*b*) Using Problem 4-26, prove the *Fundamental Theorem of Algebra:* Every polynomial $z^n + a_1 z^{n-1} + \cdots + a_n$ with $a_i \in \mathbf{C}$ has a root in $\mathbf{C}$.

4-29. If ω is a 1-form $f\,dx$ on $[0,1]$ with $f(0) = f(1)$, show that there is a unique number λ such that $\omega - \lambda\,dx = dg$ for some function g with $g(0) = g(1)$. *Hint:* Integrate $\omega - \lambda\,dx = dg$ on $[0,1]$ to find λ.

4-30. If ω is a 1-form on $\mathbf{R}^2 - 0$ such that $d\omega = 0$, prove that

$$\omega = \lambda\,d\theta + dg$$

for some $\lambda \in \mathbf{R}$ and $g\colon \mathbf{R}^2 - 0 \to \mathbf{R}$. *Hint:* If

$$c_{R,1}{}^*(\omega) = \lambda_R\,dx + d(g_R),$$

show that all numbers λ_R have the same value λ.

4-31. If $\omega \neq 0$, show that there is a chain c such that $\int_c \omega \neq 0$. Use this fact, Stokes' theorem and $\partial^2 = 0$ to prove $d^2 = 0$.

4-32. (a) Let c_1, c_2 be singular 1-cubes in $\mathbf{R}^2$ with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$. Show that there is a singular 2-cube c such that $\partial c = c_1 - c_2 + c_3 - c_4$, where c_3 and c_4 are *degenerate*, that is, $c_3([0,1])$ and $c_4([0,1])$ are points. Conclude that $\int_{c_1}\omega = \int_{c_2}\omega$ if ω is exact. Give a counterexample on $\mathbf{R}^2 - 0$ if ω is merely closed.

(b) If ω is a 1-form on a subset of $\mathbf{R}^2$ and $\int_{c_1}\omega = \int_{c_2}\omega$ for all c_1, c_2 with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$, show that ω is exact. *Hint:* Consider Problems 2-21 and 3-34.

4-33. (*A first course in complex variables.*) If $f\colon \mathbf{C} \to \mathbf{C}$, define f to be ***differentiable*** at $z_0 \in \mathbf{C}$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. (This quotient involves two complex numbers and this definition is completely different from the one in Chapter 2.) If f is differentiable at every point z in an open set A and f' is continuous on A, then f is called ***analytic*** on A.

(a) Show that $f(z) = z$ is analytic and $f(z) = \bar{z}$ is not (where $\overline{x + iy} = x - iy$). Show that the sum, product, and quotient of analytic functions are analytic.

(b) If $f = u + iv$ is analytic on A, show that u and v satisfy the *Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}.$$

Hint: Use the fact that $\lim_{z \to z_0} [f(z) - f(z_0)]/(z - z_0)$ must be the same for $z = z_0 + (x + i \cdot 0)$ and $z = z_0 + (0 + i \cdot y)$ with $x,y \to 0$. (The converse is also true, if u and v are continuously differentiable; this is more difficult to prove.)

(c) Let $T: \mathbf{C} \to \mathbf{C}$ be a linear transformation (where $\mathbf{C}$ is considered as a vector space over $\mathbf{R}$). If the matrix of T with respect to the basis $(1,i)$ is $\begin{pmatrix} a,b \\ c,d \end{pmatrix}$ show that T is multiplication by a complex number if and only if $a = d$ and $b = -c$. Part (b) shows that an analytic function $f: \mathbf{C} \to \mathbf{C}$, considered as a function $f: \mathbf{R}^2 \to \mathbf{R}^2$, has a derivative $Df(z_0)$ which is multiplication by a complex number. What complex number is this?

(d) Define

$$d(\omega + i\eta) = d\omega + i\, d\eta,$$

$$\int_c \omega + i\eta = \int_c \omega + i \int_c \eta,$$

$$(\omega + i\eta) \wedge (\theta + i\lambda) = \omega \wedge \theta - \eta \wedge \lambda + i(\eta \wedge \theta + \omega \wedge \lambda),$$

and

$$dz = dx + i\, dy.$$

Show that $d(f \cdot dz) = 0$ if and only if f satisfies the Cauchy-Riemann equations.

(e) Prove the *Cauchy Integral Theorem:* If f is analytic on A, then $\int_c f\, dz = 0$ for every closed curve c (singular 1-cube with $c(0) = c(1)$) such that $c = \partial c'$ for some 2-chain c' in A.

(f) Show that if $g(z) = 1/z$, then $g \cdot dz$ [or $(1/z)dz$ in classical notation] equals $i\, d\theta + dh$ for some function $h: \mathbf{C} - 0 \to \mathbf{R}$. Conclude that $\int_{c_{R,n}} (1/z)dz = 2\pi i n$.

(g) If f is analytic on $\{z: |z| < 1\}$, use the fact that $g(z) = f(z)/z$ is analytic in $\{z: 0 < |z| < 1\}$ to show that

$$\int_{c_{R_1,n}} \frac{f(z)}{z}\, dz = \int_{c_{R_2,n}} \frac{f(z)}{z}\, dz$$

if $0 < R_1,\ R_2 < 1$. Use (f) to evaluate $\lim_{R \to 0} \int_{c_{R,n}} f(z)/z\, dz$ and conclude:

Cauchy Integral Formula: If f is analytic on $\{z: |z| < 1\}$ and c is a closed curve in $\{z: 0 < |z| < 1\}$ with winding number n around 0, then

$$n \cdot f(0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z}\, dz.$$

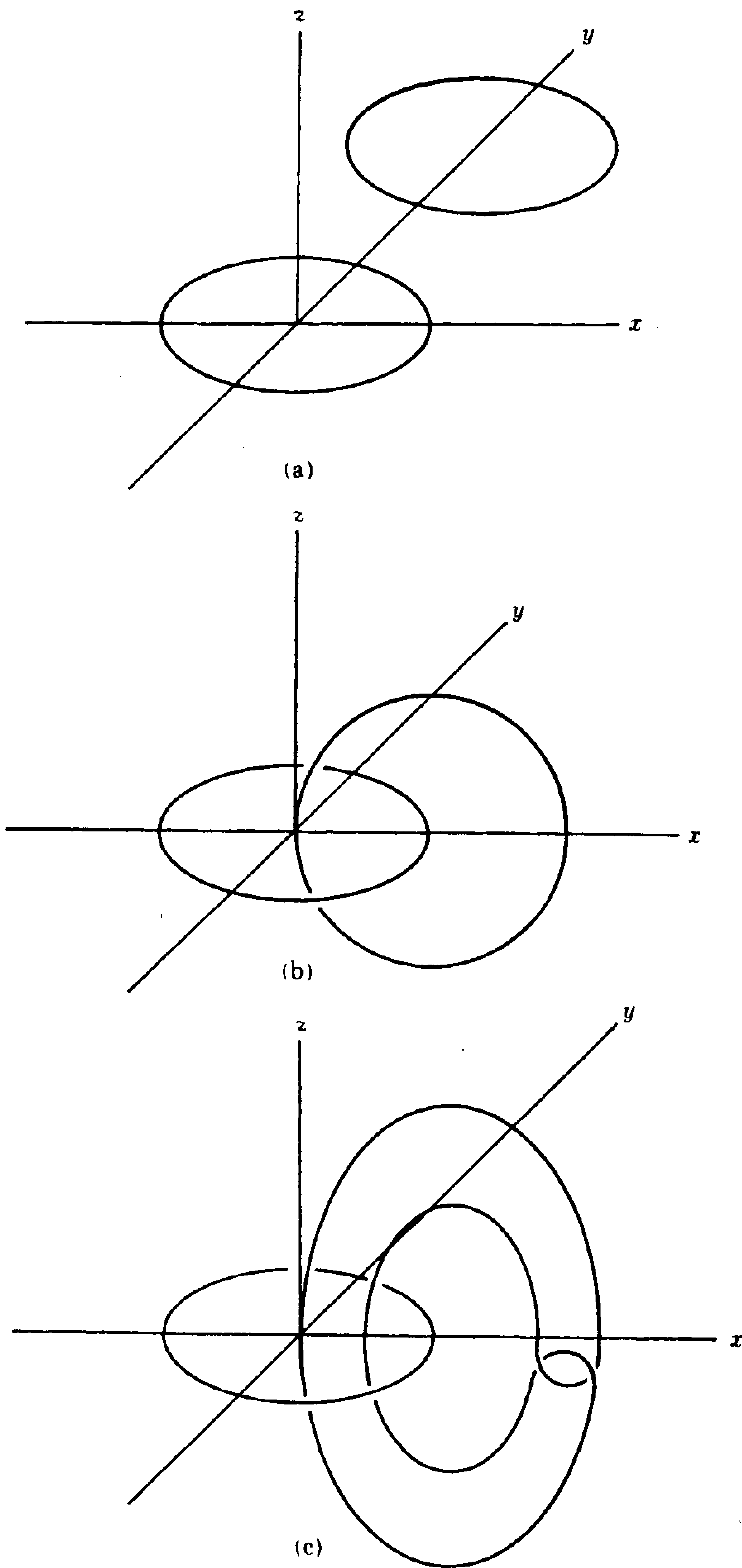


FIGURE 4-6

4-34. If $F\colon [0,1]^2 \to \mathbf{R}^3$ and $s \in [0,1]$ define $F_s\colon [0,1] \to \mathbf{R}^3$ by $F_s(t) = F(s,t)$. If each F_s is a closed curve, F is called a ***homotopy*** between the closed curve F_0 and the closed curve F_1. Suppose F and G are homotopies of closed curves; if for each s the closed curves F_s and G_s do not intersect, the pair (F,G) is called a homotopy between the nonintersecting closed curves F_0, G_0 and F_1, G_1. It is intuitively obvious that there is no such homotopy with F_0, G_0 the pair of curves shown in Figure 4-6 (a), and F_1, G_1 the pair of (b) or (c). The present problem, and Problem 5-33 prove this for (b) but the proof for (c) requires different techniques.

(a) If f, $g\colon [0,1] \to \mathbf{R}^3$ are nonintersecting closed curves define $c_{f,g}\colon [0,1]^2 \to \mathbf{R}^3 - 0$ by

$$c_{f,g}(u,v) = f(u) - g(v).$$

If (F,G) is a homotopy of nonintersecting closed curves define $C_{F,G}\colon [0,1]^3 \to \mathbf{R}^3 - 0$ by

$$C_{F,G}(s,u,v) = c_{F_s,G_s}(u,v) = F(s,u) - G(s,v).$$

Show that

$$\partial C_{F,G} = c_{F_0,G_0} - c_{F_1,G_1}.$$

(b) If ω is a closed 2-form on $\mathbf{R}^3 - 0$ show that

$$\int_{c_{F_0,G_0}} \omega = \int_{c_{F_1,G_1}} \omega.$$

5

Integration on Manifolds

MANIFOLDS

If U and V are open sets in $\mathbf{R}^n$, a differentiable function $h\colon U \to V$ with a differentiable inverse $h^{-1}\colon V \to U$ will be called a **diffeomorphism.** ("Differentiable" henceforth means "C^∞".)

A subset M of $\mathbf{R}^n$ is called a ***k*-dimensional manifold** (in $\mathbf{R}^n$) if for every point $x \in M$ the following condition is satisfied:

(M) There is an open set U containing x, an open set $V \subset \mathbf{R}^n$, and a diffeomorphism $h\colon U \to V$ such that

$$h(U \cap M) = V \cap (\mathbf{R}^k \times \{0\})$$
$$= \{y \in V : y^{k+1} = \cdots = y^n = 0\}.$$

In other words, $U \cap M$ is, "up to diffeomorphism," simply $\mathbf{R}^k \times \{0\}$ (see Figure 5-1). The two extreme cases of our definition should be noted: a point in $\mathbf{R}^n$ is a 0-dimensional manifold, and an open subset of $\mathbf{R}^n$ is an n-dimensional manifold.

One common example of an n-dimensional manifold is the

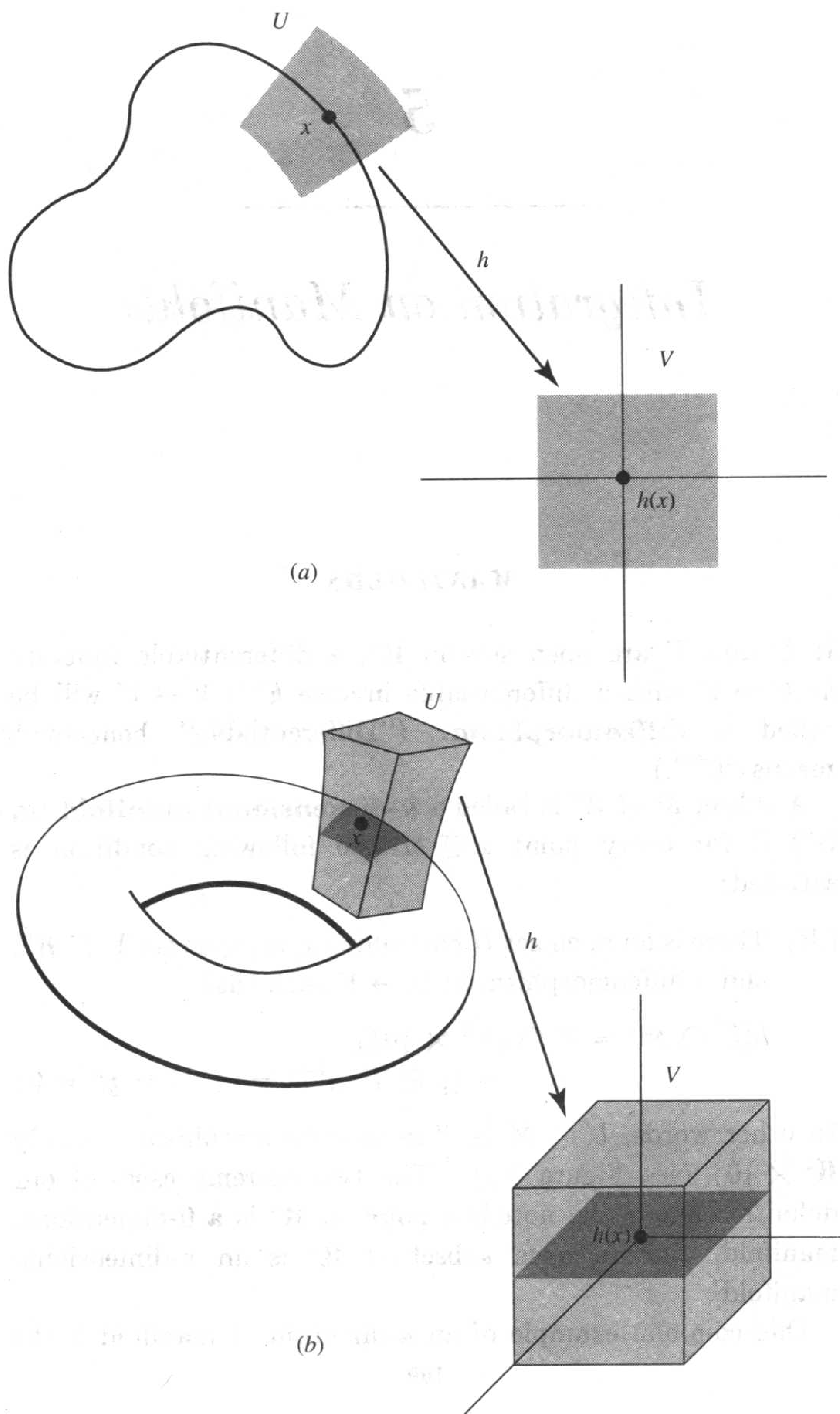


FIGURE 5-1. *A one-dimensional manifold in* $\mathbf{R}^2$ *and a two-dimensional manifold in* $\mathbf{R}^3$.

n-sphere S^n, defined as $\{x \in \mathbf{R}^{n+1} : |x| = 1\}$. We leave it as an exercise for the reader to prove that condition (M) is satisfied. If you are unwilling to trouble yourself with the details, you may instead use the following theorem, which provides many examples of manifolds (note that $S^n = g^{-1}(0)$, where $g: \mathbf{R}^{n+1} \to \mathbf{R}$ is defined by $g(x) = |x|^2 - 1$).

5-1 Theorem. *Let $A \subset \mathbf{R}^n$ be open and let $g: A \to \mathbf{R}^p$ be a differentiable function such that $g'(x)$ has rank p whenever $g(x) = 0$. Then $g^{-1}(0)$ is an $(n - p)$-dimensional manifold in $\mathbf{R}^n$.*

Proof. This follows immediately from Theorem 2-13. ▮

There is an alternative characterization of manifolds which is very important.

5-2 Theorem. *A subset M of $\mathbf{R}^n$ is a k-dimensional manifold if and only if for each point $x \in M$ the following "coordinate condition" is satisfied:*

(C) There is an open set U containing x, an open set $W \subset \mathbf{R}^k$, and a 1-1 differentiable function $f: W \to \mathbf{R}^n$ such that

(1) $f(W) = M \cap U$,
(2) $f'(y)$ has rank k for each $y \in W$,
(3) $f^{-1}: f(W) \to W$ is continuous.

[Such a function f is called **a coordinate system** around x (see Figure 5-2).]

Proof. If M is a k-dimensional manifold in $\mathbf{R}^n$, choose $h: U \to V$ satisfying (M). Let $W = \{a \in \mathbf{R}^k : (a,0) \in h(M)\}$ and define $f: W \to \mathbf{R}^n$ by $f(a) = h^{-1}(a,0)$. Clearly $f(W) = M \cap U$ and f^{-1} is continuous. If $H: U \to \mathbf{R}^k$ is $H(z) = (h^1(z), \ldots, h^k(z))$, then $H(f(y)) = y$ for all $y \in W$; therefore $H'(f(y)) \cdot f'(y) = I$ and $f'(y)$ must have rank k.

Suppose, conversely, that $f: W \to \mathbf{R}^n$ satisfies condition (C). Let $x = f(y)$. Assume that the matrix $(D_j f^i(y))$, $1 \leq i, j \leq k$ has a non-zero determinant. Define $g: W \times \mathbf{R}^{n-k} \to \mathbf{R}^n$ by

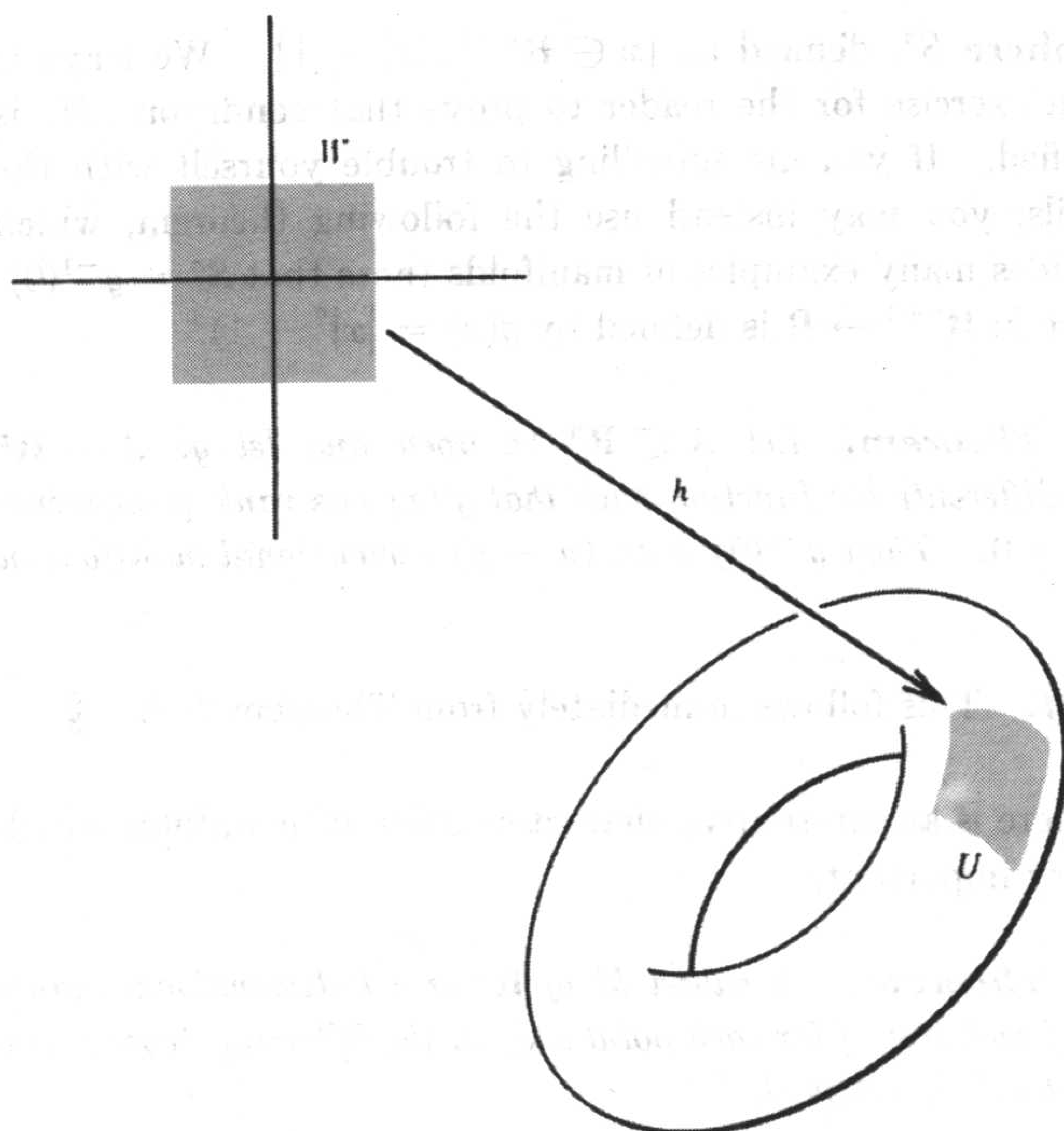


FIGURE 5-2

$g(a,b) = f(a) + (0,b)$. Then $\det g'(a,b) = \det (D_jf^i(a))$, so $\det g'(y,0) \neq 0$. By Theorem 2-11 there is an open set V_1' containing $(y,0)$ and an open set V_2' containing $g(y,0) = x$ such that $g\colon V_1' \to V_2'$ has a differentiable inverse $h\colon V_2' \to V_1'$. Since f^{-1} is continuous, $\{f(a)\colon (a,0) \in V_1'\} = U \cap f(W)$ for some open set U. Let $V_2 = V_2' \cap U$ and $V_1 = g^{-1}(V_2)$. Then $V_2 \cap M$ is exactly $\{f(a)\colon (a,0) \in V_1\} = \{g(a,0)\colon (a,0) \in V_1\}$, so

$$\begin{aligned} h(V_2 \cap M) = g^{-1}(V_2 \cap M) &= g^{-1}(\{g(a,0)\colon (a,0) \in V_1\}) \\ &= V_1 \cap (\mathbf{R}^k \times \{0\}). \quad \blacksquare \end{aligned}$$

One consequence of the proof of Theorem 5-2 should be noted. If $f_1\colon W_1 \to \mathbf{R}^n$ and $f_2\colon W_2 \to \mathbf{R}^n$ are two coordinate

systems, then

$$f_2^{-1} \circ f_1 \colon f_1^{-1}(f_2(W_2)) \to \mathbf{R}^k$$

is differentiable with non-singular Jacobian. If fact, $f_2^{-1}(y)$ consists of the first k components of $h(y)$.

The **half-space** $\mathbf{H}^k \subset \mathbf{R}^k$ is defined as $\{x \in \mathbf{R}^k \colon x^k \geq 0\}$. A subset M of $\mathbf{R}^n$ is a **k-dimensional manifold-with-boundary** (Figure 5-3) if for every point $x \in M$ either condition (M) or the following condition is satisfied:

(M') There is an open set U containing x, an open set $V \subset \mathbf{R}^n$, and a diffeomorphism $h \colon U \to V$ such that

$$\begin{aligned} h(U \cap M) &= V \cap (\mathbf{H}^k \times \{0\}) \\ &= \{y \in V \colon y^k \geq 0 \text{ and } y^{k+1} = \cdots = y^n = 0\} \end{aligned}$$

and $h(x)$ has kth component $= 0$.

It is important to note that conditions (M) and (M') cannot both hold for the same x. In fact, if $h_1 \colon U_1 \to V_1$ and $h_2 \colon U_2 \to V_2$ satisfied (M) and (M'), respectively, then $h_2 \circ h_1{}^{-1}$ would be a differentiable map that takes an open set in $\mathbf{R}^k$, containing $h(x)$, into a subset of $\mathbf{H}^k$ which is not open in $\mathbf{R}^k$. Since $\det (h_2 \circ h_1{}^{-1})' \neq 0$, this contradicts Problem 2-36. The set of all points $x \in M$ for which condition M' is satisfied is called the **boundary** of M and denoted ∂M. This

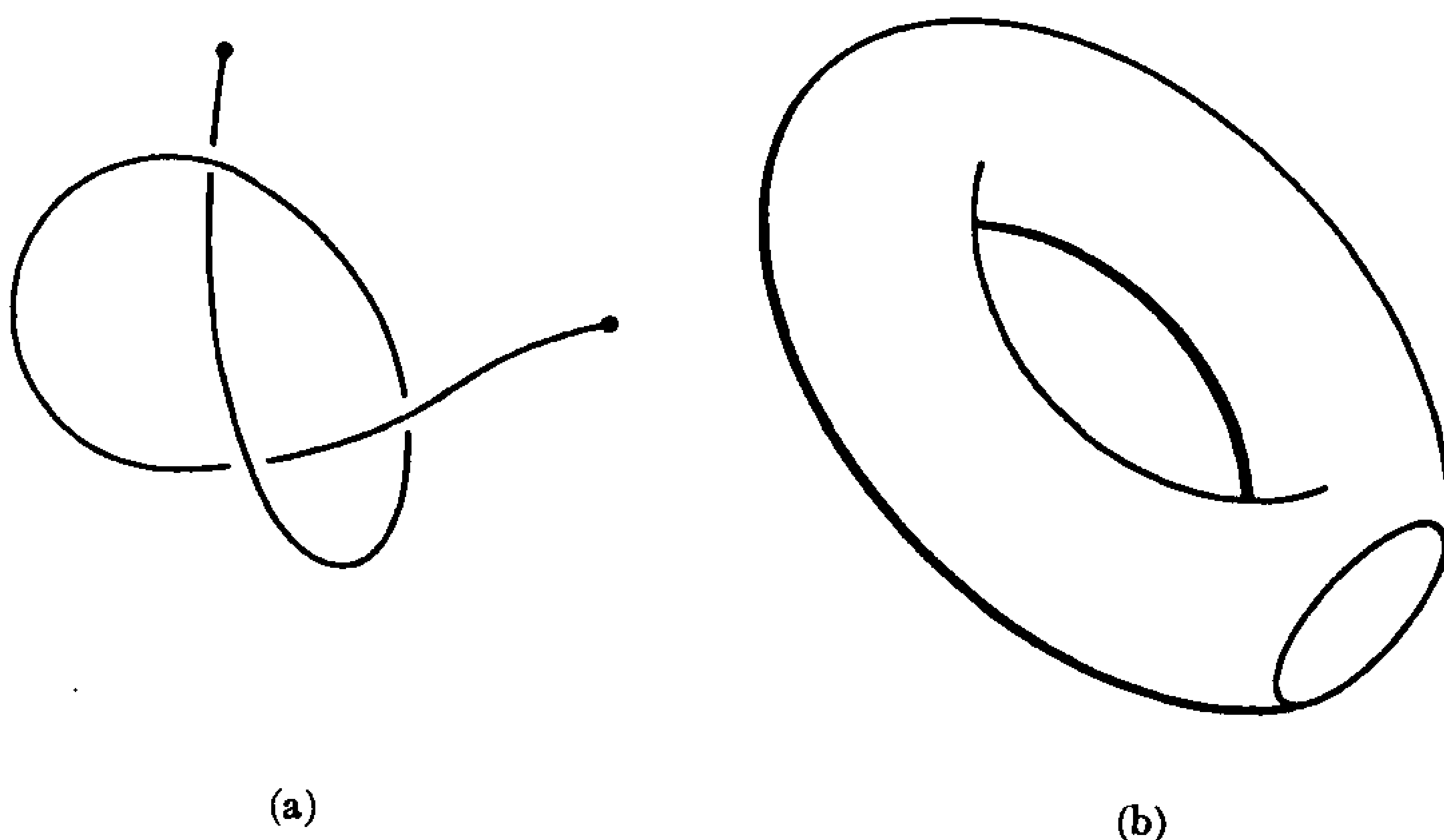


FIGURE 5-3. A one-dimensional and a two-dimensional manifold-with-boundary in $\mathbf{R}^3$.

must not be confused with the boundary of a set, as defined in Chapter 1 (see Problems 5-3 and 5-8).

Problems. **5-1.** If M is a k-dimensional manifold-with-boundary, prove that ∂M is a $(k - 1)$-dimensional manifold and $M - \partial M$ is a k-dimensional manifold.

5-2. Find a counterexample to Theorem 5-2 if condition (3) is omitted. *Hint:* Wrap an open interval into a figure six.

5-3. (a) Let $A \subset \mathbf{R}^n$ be an open set such that boundary A is an $(n - 1)$-dimensional manifold. Show that $N = A \cup$ boundary A is an n-dimensional manifold-with-boundary. (It is well to bear in mind the following example: if $A = \{x \in \mathbf{R}^n: |x| < 1 \text{ or } 1 < |x| < 2\}$ then $N = A \cup$ boundary A is a manifold-with-boundary, but $\partial N \neq$ boundary A.)

(b) Prove a similar assertion for an open subset of an n-dimensional manifold.

5-4. Prove a partial converse of Theorem 5-1: If $M \subset \mathbf{R}^n$ is a k-dimensional manifold and $x \in M$, then there is an open set $A \subset \mathbf{R}^n$ containing x and a differentiable function $g: A \to \mathbf{R}^{n-k}$ such that $A \cap M = g^{-1}(0)$ and $g'(y)$ has rank $n - k$ when $g(y) = 0$.

5-5. Prove that a k-dimensional (vector) subspace of $\mathbf{R}^n$ is a k-dimensional manifold.

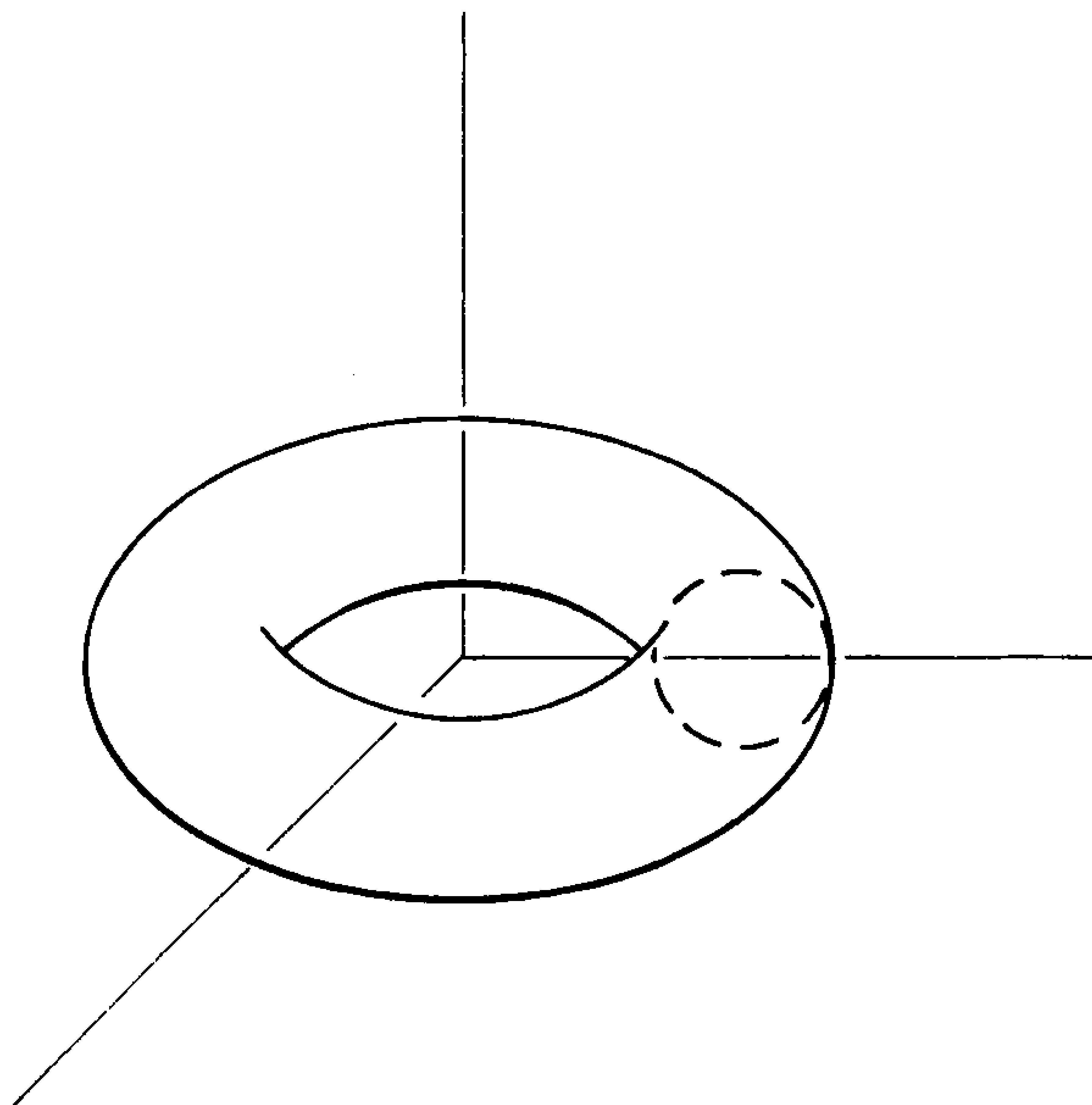


FIGURE 5-4

5-6. If $f: \mathbf{R}^n \to \mathbf{R}^m$, the **graph** of f is $\{(x,y): y = f(x)\}$. Show that the graph of f is an n-dimensional manifold if and only if f is differentiable.

5-7. Let $\mathbf{K}^n = \{x \in \mathbf{R}^n: x^1 = 0 \text{ and } x^2, \ldots, x^{n-1} > 0\}$. If $M \subset \mathbf{K}^n$ is a k-dimensional manifold and N is obtained by revolving M around the axis $x^1 = \cdots = x^{n-1} = 0$, show that N is a $(k+1)$-dimensional manifold. Example: the torus (Figure 5-4).

5-8. (a) If M is a k-dimensional manifold in $\mathbf{R}^n$ and $k < n$, show that M has measure 0.

(b) If M is a closed n-dimensional manifold-with-boundary in $\mathbf{R}^n$, show that the boundary of M is ∂M. Give a counterexample if M is not closed.

(c) If M is a compact n-dimensional manifold-with-boundary in $\mathbf{R}^n$, show that M is Jordan-measurable.

FIELDS AND FORMS ON MANIFOLDS

Let M be a k-dimensional manifold in $\mathbf{R}^n$ and let $f: W \to \mathbf{R}^n$ be a coordinate system around $x = f(a)$. Since $f'(a)$ has rank k, the linear transformation $f_*: \mathbf{R}^k{}_a \to \mathbf{R}^n{}_x$ is 1-1, and $f_*(\mathbf{R}^k{}_a)$ is a k-dimensional subspace of $\mathbf{R}^n{}_x$. If $g: V \to \mathbf{R}^n$ is another coordinate system, with $x = g(b)$, then

$$g_*(\mathbf{R}^k{}_b) = f_*(f^{-1} \circ g)_*(\mathbf{R}^k{}_b) = f_*(\mathbf{R}^k{}_a).$$

Thus the k-dimensional subspace $f_*(\mathbf{R}^k{}_a)$ does not depend on the coordinate system f. This subspace is denoted M_x, and is called the **tangent space** of M at x (see Figure 5-5). In later sections we will use the fact that there is a natural inner product T_x on M_x, induced by that on $\mathbf{R}^n{}_x$: if $v,w \in M_x$ define $T_x(v,w) = \langle v,w \rangle_x$.

Suppose that A is an open set containing M, and F is a differentiable vector field on A such that $F(x) \in M_x$ for each $x \in M$. If $f: W \to \mathbf{R}^n$ is a coordinate system, there is a unique (differentiable) vector field G on W such that $f_*(G(a)) = F(f(a))$ for each $a \in W$. We can also consider a function F which merely assigns a vector $F(x) \in M_x$ for each $x \in M$; such a function is called **a vector field on** M. There is still a unique vector field G on W such that $f_*(G(a)) = F(f(a))$ for $a \in W$; we *define* F to be differentiable if G is differentiable. Note that our definition does not depend on the coordinate

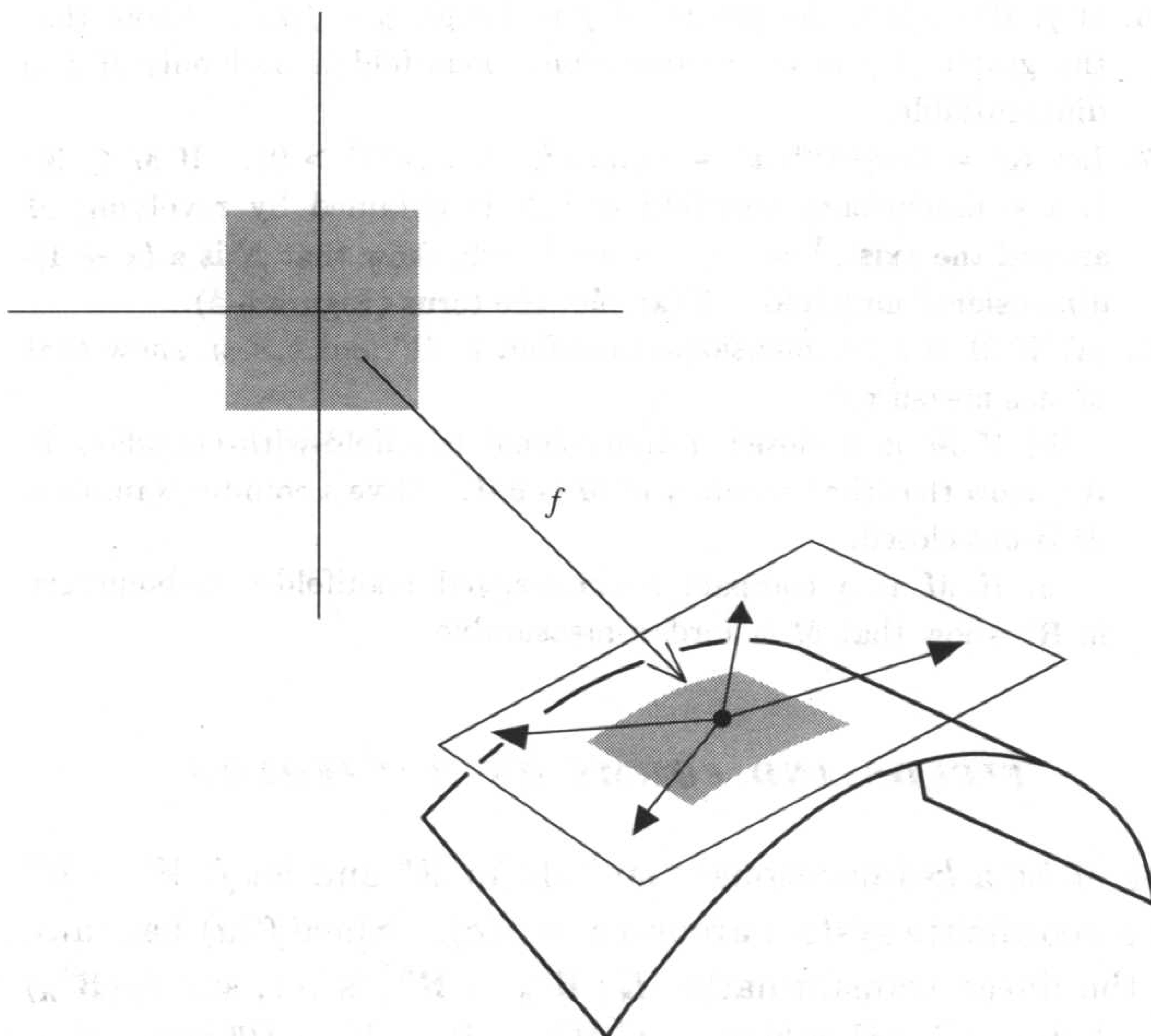


FIGURE 5-5

system chosen: if $g\colon V \to \mathbf{R}^n$ and $g_*(H(b)) = F(g(b))$ for all $b \in V$, then the component functions of $H(b)$ must equal the component functions of $G(f^{-1}(g(b)))$, so H is differentiable if G is.

Precisely the same considerations hold for forms. A function ω which assigns $\omega(x) \in \Lambda^p(M_x)$ for each $x \in M$ is called a **p-form on M**. If $f\colon W \to \mathbf{R}^n$ is a coordinate system, then $f^*\omega$ is a p-form on W; we *define* ω to be differentiable if $f^*\omega$ is. A p-form ω on M can be written as

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1,\ldots,i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Here the functions $\omega_{i_1,\ldots,i_p}$ are defined only on M. The definition of $d\omega$ given previously would make no sense here, since $D_j(\omega_{i_1,\ldots,i_p})$ has no meaning. Nevertheless, there is a reasonable way of defining $d\omega$.

5-3 Theorem. *There is a unique $(p+1)$-form $d\omega$ on M such that for every coordinate system $f\colon W \to \mathbf{R}^n$ we have*

$$f^*(d\omega) = d(f^*\omega).$$

Proof. If $f\colon W \to \mathbf{R}^n$ is a coordinate system with $x = f(a)$ and $v_1, \ldots, v_{p+1} \in M_x$, there are unique $w_1, \ldots, w_{p+1}$ in $\mathbf{R}^k{}_a$ such that $f_*(w_i) = v_i$. Define $d\omega(x)(v_1, \ldots, v_{p+1}) = d(f^*\omega)(a)(w_1, \ldots, w_{p+1})$. One can check that this definition of $d\omega(x)$ does not depend on the coordinate system f, so that $d\omega$ is well-defined. Moreover, it is clear that $d\omega$ has to be defined this way, so $d\omega$ is unique. ▌

It is often necessary to choose an orientation μ_x for each tangent space M_x of a manifold M. Such choices are called **consistent** (Figure 5-6) provided that for every coordinate

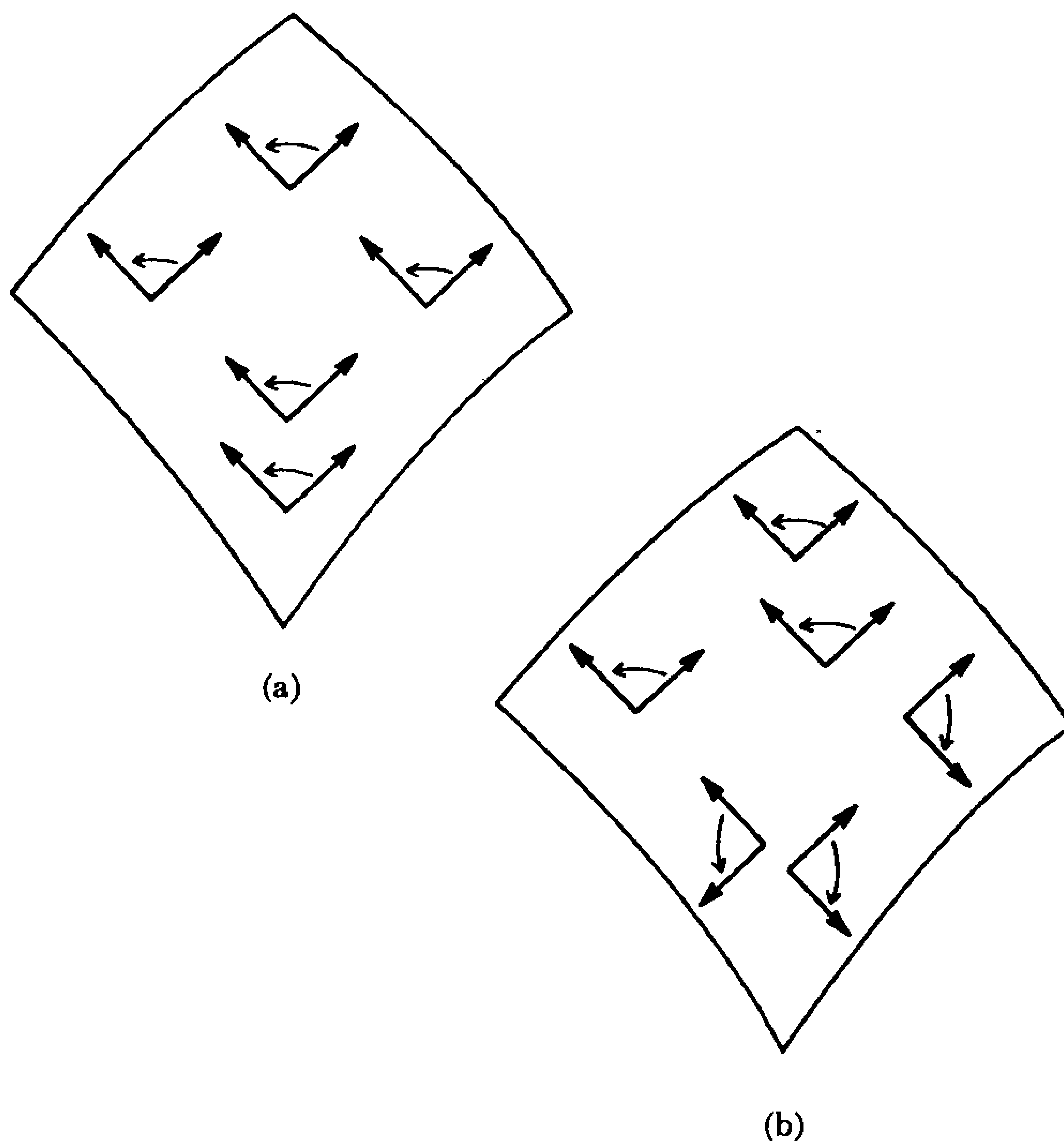


FIGURE 5-6. *(a) Consistent and (b) inconsistent choices of orientations.*

system $f\colon W \to \mathbf{R}^n$ and $a,b \in W$ the relation

$$[f_*((e_1)_a), \ldots ,f_*((e_k)_a)] = \mu_{f(a)}$$

holds if and only if

$$[f_*((e_1)_b), \ldots ,f_*((e_k)_b)] = \mu_{f(b)}.$$

Suppose orientations μ_x have been chosen consistently. If $f\colon W \to \mathbf{R}^n$ is a coordinate system such that

$$[f_*((e_1)_a), \ldots , f_*((e_k)_a)] = \mu_{f(a)}$$

for one, and hence for every $a \in W$, then f is called **orientation-preserving.** If f is *not* orientation-preserving and $T\colon \mathbf{R}^k \to \mathbf{R}^k$ is a linear transformation with $\det T = -1$, then $f \circ T$ *is* orientation-preserving. Therefore there is an orientation-preserving coordinate system around each point. If f and g are orientation-preserving and $x = f(a) = g(b)$, then the relation

$$[f_*((e_1)_a), \ldots ,f_*((e_k)_a)] = \mu_x = [g_*((e_1)_b), \ldots ,g_*((e_k)_b)]$$

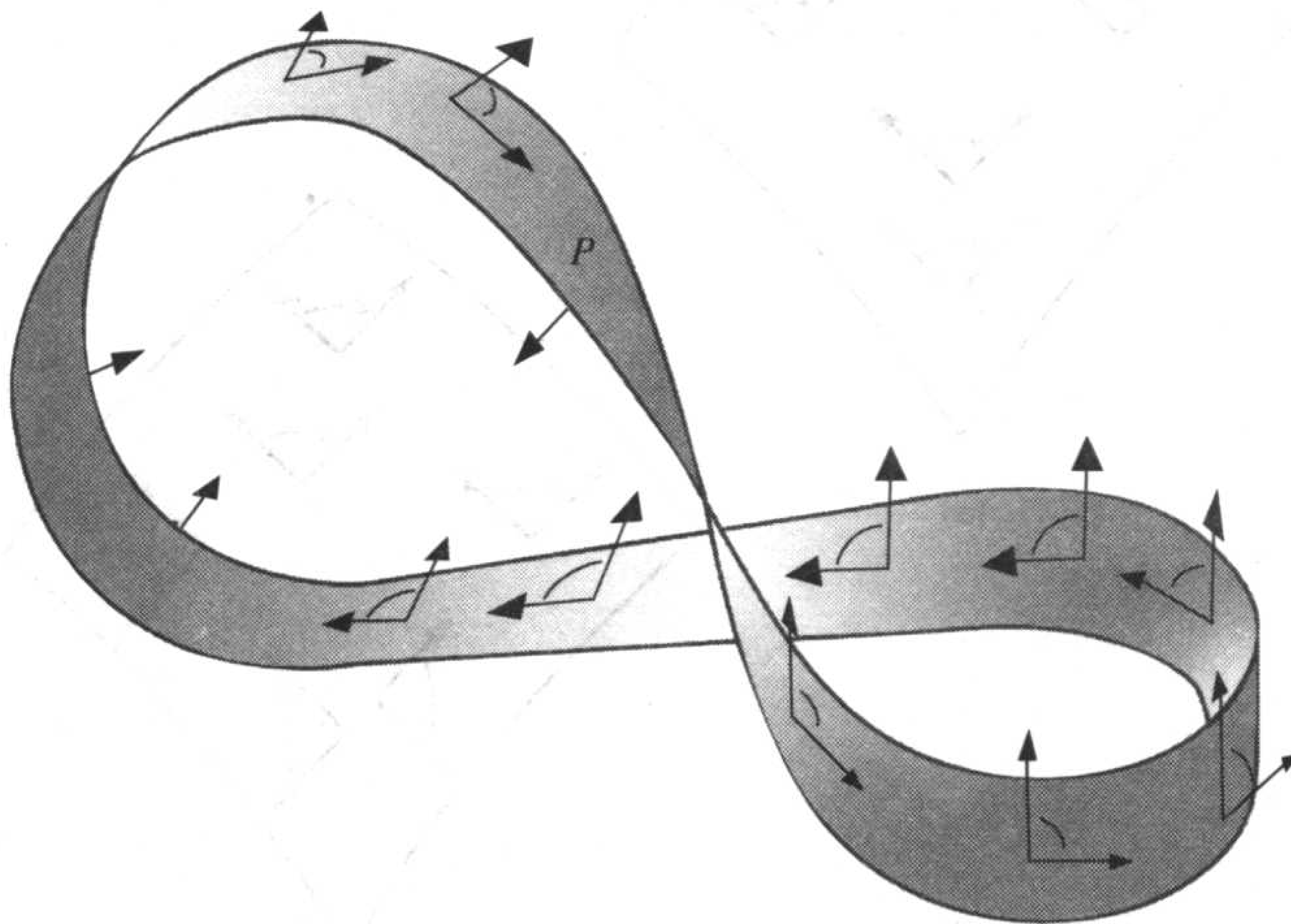


FIGURE 5-7. *The Möbius strip, a non-orientable manifold. A basis begins at P, moves to the right and around, and comes back to P with the wrong orientation.*

implies that

$$[(g^{-1}\circ f)_*((e_1)_a), \ldots, (g^{-1}\circ f)_*((e_k)_a)] = [(e_1)_b, \ldots, (e_k)_b],$$

so that det $(g^{-1}\circ f)' > 0$, an important fact to remember.

A manifold for which orientations μ_x can be chosen consistently is called **orientable,** and a particular choice of the μ_x is called an **orientation** μ of M. A manifold together with an orientation μ is called an **oriented** manifold. The classical example of a non-orientable manifold is the Möbius strip. A model can be made by gluing together the ends of a strip of paper which has been given a half twist (Figure 5-7).

Our definitions of vector fields, forms, and orientations can be made for manifolds-with-boundary also. If M is a k-dimensional manifold-with-boundary and $x \in \partial M$, then $(\partial M)_x$ is a $(k-1)$-dimensional subspace of the k-dimensional vector space M_x. Thus there are exactly two unit vectors in M_x which are perpendicular to $(\partial M)_x$; they can be distinguished as follows (Figure 5-8). If $f\colon W \to \mathbf{R}^n$ is a coordinate system with $W \subset H^k$ and $f(0) = x$, then only one of these unit vectors is $f_*(v_0)$ for some v_0 with $v^k < 0$. This unit vector is called the **outward unit normal** $n(x)$; it is not hard to check that this definition does not depend on the coordinate system f.

Suppose that μ is an orientation of a k-dimensional manifold-with-boundary M. If $x \in \partial M$, choose $v_1, \ldots, v_{k-1} \in (\partial M)_x$ so that $[n(x), v_1, \ldots, v_{k-1}] = \mu_x$. If it is also true that $[n(x), w_1, \ldots, w_{k-1}] = \mu_x$, then both $[v_1, \ldots, v_{k-1}]$ and $[w_1, \ldots, w_{k-1}]$ are the same orientation for $(\partial M)_x$. This orientation is denoted $(\partial\mu)_x$. It is easy to see that the orientations $(\partial\mu)_x$, for $x \in \partial M$, are consistent on ∂M. Thus if M is orientable, ∂M is also orientable, and an orientation μ for M determines an orientation $\partial\mu$ for ∂M, called the **induced orientation.** If we apply these definitions to $\mathbf{H}^k$ with the usual orientation, we find that the induced orientation on $\mathbf{R}^{k-1} = \{x \in \mathbf{H}^k\colon x^k = 0\}$ is $(-1)^k$ times the usual orientation. The reason for such a choice will become clear in the next section.

If M is an *oriented* $(n-1)$-dimensional manifold in $\mathbf{R}^n$, a substitute for outward unit normal vectors can be defined,

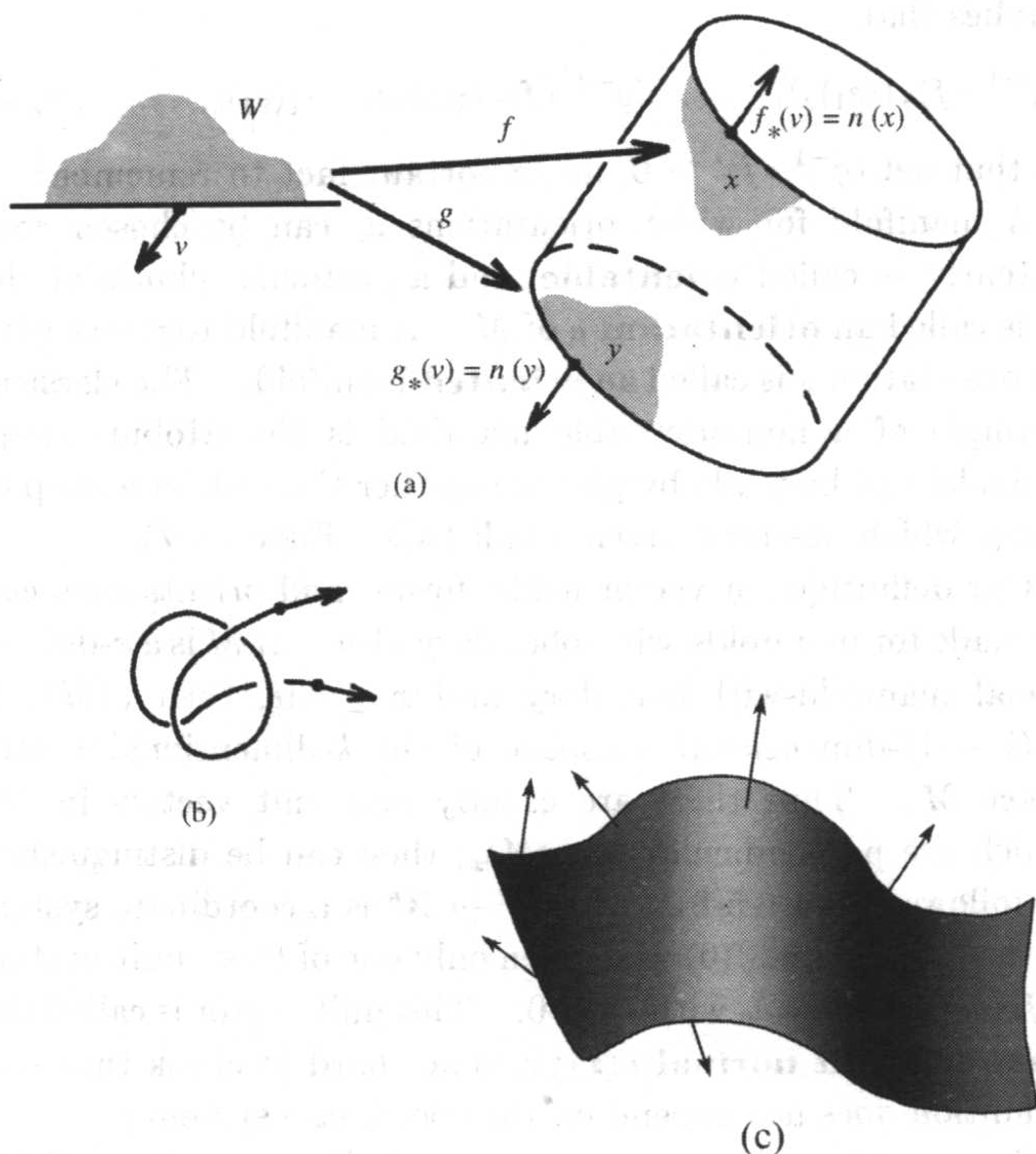


FIGURE 5-8. *Some outward unit normal vectors of manifolds-with-boundary in* $\mathbf{R}^3$.

even though M is not necessarily the boundary of an n-dimensional manifold. If $[v_1, \ldots, v_{n-1}] = \mu_x$, we choose $n(x)$ in $\mathbf{R}^n{}_x$ so that $n(x)$ is a unit vector perpendicular to M_x and $[n(x), v_1, \ldots, v_{n-1}]$ is the usual orientation of $\mathbf{R}^n{}_x$. We still call $n(x)$ the outward unit normal to M (determined by μ). The vectors $n(x)$ vary continuously on M, in an obvious sense. Conversely, if a continuous family of unit normal vectors $n(x)$ is defined on all of M, then we can determine an orientation of M. This shows that such a continuous choice of normal vectors is impossible on the Möbius strip. In the paper model of the Möbius strip the two sides of the paper (which has thickness) may be thought of as the end points of the unit

normal vectors in both directions. The impossibility of choosing normal vectors continuously is reflected by the famous property of the paper model. The paper model is one-sided (if you start to paint it on one side you end up painting it all over); in other words, choosing $n(x)$ arbitrarily at one point, and then by the continuity requirement at other points, eventually forces the opposite choice for $n(x)$ at the initial point.

Problems. **5-9.** Show that M_x consists of the tangent vectors at t of curves c in M with $c(t) = x$.

5-10. Suppose $\mathcal{C}$ is a collection of coordinate systems for M such that (1) For each $x \in M$ there is $f \in \mathcal{C}$ which is a coordinate system around x; (2) if $f,g \in \mathcal{C}$, then $\det (f^{-1} \circ g)' > 0$. Show that there is a unique orientation of M such that f is orientation-preserving if $f \in \mathcal{C}$.

5-11. If M is an n-dimensional manifold-with-boundary in $\mathbf{R}^n$, define μ_x as the usual orientation of $M_x = \mathbf{R}^n{}_x$ (the orientation μ so defined is the **usual orientation** of M). If $x \in \partial M$, show that the two definitions of $n(x)$ given above agree.

5-12. (a) If F is a differentiable vector field on $M \subset \mathbf{R}^n$, show that there is an open set $A \supset M$ and a differentiable vector field $\tilde{F}$ on A with $\tilde{F}(x) = F(x)$ for $x \in M$. *Hint:* Do this locally and use partitions of unity.

(b) If M is closed, show that we can choose $A = \mathbf{R}^n$.

5-13. Let $g: A \to R^p$ be as in Theorem 5-1.

(a) If $x \in M = g^{-1}(0)$, let $h: U \to R^n$ be the essentially unique diffeomorphism such that $g \circ h(y) = (y^{n-p+1}, \ldots, y^n)$ and $h(0) = x$. Define $f: R^{n-p} \to R^n$ by $f(a) = h(0,a)$. Show that f_* is 1-1 so that the $n - p$ vectors $f_*((e_1)_0), \ldots, f_*((e_{n-p})_0)$ are linearly independent.

(b) Show that orientations μ_x can be defined consistently, so that M is orientable.

(c) If $p = 1$, show that the components of the outward normal at x are some multiple of $D_1g(x), \ldots, D_ng(x)$.

5-14. If $M \subset \mathbf{R}^n$ is an orientable $(n - 1)$-dimensional manifold, show that there is an open set $A \subset \mathbf{R}^n$ and a differentiable $g: A \to \mathbf{R}^1$ so that $M = g^{-1}(0)$ and $g'(x)$ has rank 1 for $x \in M$. *Hint:* Problem 5-4 does this locally. Use the orientation to choose consistent local solutions and use partitions of unity.

5-15. Let M be an $(n - 1)$-dimensional manifold in $\mathbf{R}^n$. Let $M(\varepsilon)$ be the set of end points of normal vectors (in both directions) of length ε and suppose ε is small enough so that $M(\varepsilon)$ is also an

$(n-1)$-dimensional manifold. Show that $M(\varepsilon)$ is orientable (even if M is not). What is $M(\varepsilon)$ if M is the Möbius strip?

5-16. Let $g\colon A \to \mathbf{R}^p$ be as in Theorem 5-1. If $f\colon \mathbf{R}^n \to \mathbf{R}$ is differentiable and the maximum (or minimum) of f on $g^{-1}(0)$ occurs at a, show that there are $\lambda_1, \ldots, \lambda_p \in \mathbf{R}$, such that

$$(1)\quad D_jf(a) = \sum_{i=1}^{n} \lambda_i D_j g^i(a) \qquad j = 1, \ldots, n.$$

Hint: This equation can be written $df(a) = \sum_{i=1}^{n} \lambda_i dg^i(a)$ and is obvious if $g(x) = (x^{n-p+1}, \ldots, x^n)$.

The maximum of f on $g^{-1}(0)$ is sometimes called the maximum of f subject to the **constraints** $g^i = 0$. One can attempt to find a by solving the system of equations (1). In particular, if $g\colon A \to \mathbf{R}$, we must solve $n+1$ equations

$$\begin{aligned} D_jf(a) &= \lambda D_j g(a), \\ g(a) &= 0, \end{aligned}$$

in $n+1$ unknowns $a^1, \ldots, a^n, \lambda$, which is often very simple if we leave the equation $g(a) = 0$ for last. This is **Lagrange's method,** and the useful but irrelevant λ is called a **Lagrangian multiplier.** The following problem gives a nice theoretical use for Lagrangian multipliers.

5-17. (a) Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be self-adjoint with matrix $A = (a_{ij})$, so that $a_{ij} = a_{ji}$. If $f(x) = \langle Tx, x\rangle = \sum a_{ij}x^i x^j$, show that $D_kf(x) = 2\sum_{j=1}^{n} a_{kj}x^j$. By considering the maximum of $\langle Tx, x\rangle$ on S^{n-1} show that there is $x \in S^{n-1}$ and $\lambda \in \mathbf{R}$ with $Tx = \lambda x$.

(b) If $V = \{y \in \mathbf{R}^n\colon \langle x, y\rangle = 0\}$, show that $T(V) \subset V$ and $T\colon V \to V$ is self-adjoint.

(c) Show that T has a basis of eigenvectors.

STOKES' THEOREM ON MANIFOLDS

If ω is a p-form on a k-dimensional manifold-with-boundary M and c is a singular p-cube in M, we define

$$\int_c \omega = \int_{[0,1]^p} c^*\omega$$

precisely as before; integrals over p-chains are also defined as before. In the case $p = k$ it may happen that there is an open set $W \supset [0,1]^k$ and a coordinate system $f\colon W \to \mathbf{R}^n$ such that $c(x) = f(x)$ for $x \in [0,1]^k$; a k-cube in M will always be

understood to be of this type. If M is oriented, the singular k-cube c is called **orientation-preserving** if f is.

5-4 Theorem. *If c_1,c_2: $[0,1]^k \to M$ are two* orientation-preserving *singular k-cubes in the oriented k-dimensional manifold M and ω is a k-form on M such that $\omega = 0$ outside of $c_1([0,1]^k) \cap c_2([0,1]^k)$, then*

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

Proof. We have

$$\int_{c_1} \omega = \int_{[0,1]^k} c_1{}^*(\omega) = \int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2{}^*(\omega).$$

(Here $c_2^{-1} \circ c_1$ is defined only on a subset of $[0,1]^k$ and the second equality depends on the fact that $\omega = 0$ outside of $c_1([0,1]^k) \cap c_2([0,1]^k)$.) It therefore suffices to show that

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2{}^*(\omega) = \int_{[0,1]^k} c_2{}^*(\omega) = \int_{c_2} \omega.$$

If $c_2{}^*(\omega) = f\,dx^1 \wedge \cdots \wedge dx^k$ and $c_2{}^{-1} \circ c_1$ is denoted by g, then by Theorem 4-9 we have

$$\begin{aligned}(c_2{}^{-1} \circ c_1)^* c_2{}^*(\omega) &= g^*(f\,dx^1 \wedge \cdots \wedge dx^k)\\ &= (f \circ g) \cdot \det g' \cdot dx^1 \wedge \cdots \wedge dx^k\\ &= (f \circ g) \cdot |\det g'| \cdot dx^1 \wedge \cdots \wedge dx^k,\end{aligned}$$

since $\det g' = \det(c_2{}^{-1} \circ c_1)' > 0$. The result now follows from Theorem 3-13. ∎

The last equation in this proof should help explain why we have had to be so careful about orientations.

Let ω be a k-form on an oriented k-dimensional manifold M. If there is an orientation-preserving singular k-cube c in M such that $\omega = 0$ outside of $c([0,1]^k)$, we define

$$\int_M \omega = \int_c \omega.$$

Theorem 5-4 shows $\int_M \omega$ does not depend on the choice of c.

Suppose now that ω is an arbitrary k-form on M. There is an open cover $\mathcal{O}$ of M such that for each $U \in \mathcal{O}$ there is an orientation-preserving singular k-cube c with $U \subset c([0,1]^k)$. Let Φ be a partition of unity for M subordinate to this cover. We define

$$\int_M \omega = \sum_{\varphi \in \Phi} \int_M \varphi \cdot \omega$$

provided the sum converges as described in the discussion pre ceding Theorem 3-12 (this is certainly true if M is compact). An argument similar to that in Theorem 3-12 shows that $\int_M \omega$ does not depend on the cover $\mathcal{O}$ or on Φ.

All our definitions could have been given for a k-dimensional manifold-with-boundary M with orientation μ. Let ∂M have the induced orientation $\partial\mu$. Let c be an orientation-preserving k-cube in M such that $c_{(k,0)}$ lies in ∂M and is the only face which has any interior points in ∂M. As the remarks after the definition of $\partial\mu$ show, $c_{(k,0)}$ is orientation-preserving if k is even, but not if k is odd. Thus, if ω is a $(k-1)$-form on M which is 0 outside of $c([0,1]^k)$, we have

$$\int_{c_{(k,0)}} \omega = (-1)^k \int_{\partial M} \omega.$$

On the other hand, $c_{(k,0)}$ appears with coefficient $(-1)^k$ in ∂c. Therefore

$$\int_{\partial c} \omega = \int_{(-1)^k c_{(k,0)}} \omega = (-1)^k \int_{c_{(k,0)}} \omega = \int_{\partial M} \omega.$$

Our choice of $\partial\mu$ was made to eliminate any minus signs in this equation, and in the following theorem.

5-5 *Theorem (Stokes' Theorem).* *If M is a compact oriented k-dimensional manifold-with-boundary and ω is a $(k-1)$-form on M, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

(Here ∂M is given the induced orientation.)

Proof. Suppose first that there is an orientation-preserving singular k-cube in $M - \partial M$ such that $\omega = 0$ outside of

$c([0,1]^k)$. By Theorem 4-13 and the definition of $d\omega$ we have

$$\int_c d\omega = \int_{[0,1]^k} c^*(d\omega) = \int_{[0,1]^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega.$$

Then

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = 0,$$

since $\omega = 0$ on ∂c. On the other hand, $\int_{\partial M} \omega = 0$ since $\omega = 0$ on ∂M.

Suppose next that there is an orientation-preserving singular k-cube in M such that $c_{(k,0)}$ is the only face in ∂M, and $\omega = 0$ outside of $c([0,1])^k$. Then

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = \int_{\partial M} \omega.$$

Now consider the general case. There is an open cover $\mathcal{O}$ of M and a partition of unity Φ for M subordinate to $\mathcal{O}$ such that for each $\varphi \in \Phi$ the form $\varphi \cdot \omega$ is of one of the two sorts already considered. We have

$$0 = d(1) = d\Big(\sum_{\varphi\in\Phi} \varphi\Big) = \sum_{\varphi\in\Phi} d\varphi,$$

so that

$$\sum_{\varphi\in\Phi} d\varphi \wedge \omega = 0.$$

Since M is compact, this is a finite sum and we have

$$\sum_{\varphi\in\Phi} \int_M d\varphi \wedge \omega = 0.$$

Therefore

$$\begin{aligned}\int_M d\omega &= \sum_{\varphi\in\Phi} \int_M \varphi \cdot d\omega = \sum_{\varphi\in\Phi} \int_M d\varphi \wedge \omega + \varphi \cdot d\omega \\ &= \sum_{\varphi\in\Phi} \int_M d(\varphi \cdot \omega) = \sum_{\varphi\in\Phi} \int_{\partial M} \varphi \cdot \omega \\ &= \int_{\partial M} \omega. \quad \blacksquare\end{aligned}$$

Problems. **5-18.** If M is an n-dimensional manifold (or manifold-with-boundary) in $\mathbf{R}^n$, with the usual orientation, show that

$\int_M f\, dx^1 \wedge \cdots \wedge dx^n$, as defined in this section, is the same as $\int_M f$, as defined in Chapter 3.

5-19. (a) Show that Theorem 5-5 is false if M is not compact. *Hint:* If M is a manifold-with-boundary for which 5-5 holds, then $M - \partial M$ is also a manifold-with-boundary (with empty boundary).

(b) Show that Theorem 5-5 holds for noncompact M provided that ω vanishes outside of a compact subset of M.

5-20. If ω is a $(k-1)$-form on a compact k-dimensional manifold M, prove that $\int_M d\omega = 0$. Give a counterexample if M is not compact.

5-21. An **absolute k-tensor** on V is a function $\eta: V^k \to \mathbf{R}$ of the form $|\omega|$ for $\omega \in \Lambda^k(V)$. An **absolute k-form** on M is a function η such that $\eta(x)$ is an absolute k-tensor on M_x. Show that $\int_M \eta$ can be defined, even if M is not orientable.

5-22. If $M_1 \subset \mathbf{R}^n$ is an n-dimensional manifold-with-boundary and $M_2 \subset M_1 - \partial M_1$ is an n-dimensional manifold-with-boundary, and M_1, M_2 are compact, prove that

$$\int_{\partial M_1} \omega = \int_{\partial M_2} \omega,$$

where ω is an $(n-1)$-form on M_1, and ∂M_1 and ∂M_2 have the orientations induced by the usual orientations of M_1 and M_2. *Hint:* Find a manifold-with-boundary M such that $\partial M = \partial M_1 \cup \partial M_2$ and such that the induced orientation on ∂M agrees with that for ∂M_1 on ∂M_1 and is the negative of that for ∂M_2 on ∂M_2.

THE VOLUME ELEMENT

Let M be a k-dimensional manifold (or manifold-with-boundary) in $\mathbf{R}^n$, with an orientation μ. If $x \in M$, then μ_x and the inner product T_x we defined previously determine a volume element $\omega(x) \in \Lambda^k(M_x)$. We therefore obtain a nowhere-zero k-form ω on M, which is called the **volume element** on M (determined by μ) and denoted dV, even though it is not generally the differential of a $(k-1)$-form. The **volume** of M is defined as $\int_M dV$, provided this integral exists, which is certainly the case if M is compact. "Volume" is usually called **length** or **surface area** for one- and two-dimensional manifolds, and dV is denoted ds (the "element of length") or dA [or dS] (the "element of [surface] area").

A concrete case of interest to us is the volume element of an

oriented surface (two-dimensional manifold) M in $\mathbf{R}^3$. Let $n(x)$ be the unit outward normal at $x \in M$. If $\omega \in \Lambda^2(M_x)$ is defined by

$$\omega(v,w) = \det \begin{pmatrix} v \\ w \\ n(x) \end{pmatrix},$$

then $\omega(v,w) = 1$ if v and w are an orthonormal basis of M_x with $[v,w] = \mu_x$. Thus $dA = \omega$. On the other hand, $\omega(v,w) = \langle v \times w, n(x)\rangle$ by definition of $v \times w$. Thus we have

$$dA(v,w) = \langle v \times w, n(x)\rangle.$$

Since $v \times w$ is a multiple of $n(x)$ for $v,w \in M_x$, we conclude that

$$dA(v,w) = |v \times w|$$

if $[v,w] = \mu_x$. If we wish to compute the area of M, we must evaluate $\int_{[0,1]^2} c^*(dA)$ for orientation-preserving singular 2-cubes c. Define

$$E(a) = [D_1c^1(a)]^2 + [D_1c^2(a)]^2 + [D_1c^3(a)]^2,$$

$$F(a) = D_1c^1(a) \cdot D_2c^1(a) + D_1c^2(a) \cdot D_2c^2(a) + D_1c^3(a) \cdot D_2c^3(a),$$

$$G(a) = [D_2c^1(a)]^2 + [D_2c^2(a)]^2 + [D_2c^3(a)]^2.$$

Then

$$\begin{aligned} c^*(dA)((e_1)_a,(e_2)_a) &= dA(c_*((e_1)_a),c_*((e_2)_a)) \\ &= |(D_1c^1(a),D_1c^2(a),D_1c^3(a)) \times (D_2c^1(a),D_2c^2(a),D_2c^3(a))| \\ &= \sqrt{E(a)G(a) - F(a)^2} \end{aligned}$$

by Problem 4-9. Thus

$$\int_{[0,1]^2} c^*(dA) = \int_{[0,1]^2} \sqrt{EG - F^2}.$$

Calculating surface area is clearly a foolhardy enterprise; fortunately one seldom needs to know the area of a surface. Moreover, there is a simple expression for dA which suffices for theoretical considerations.

5-6 Theorem. *Let M be an oriented two-dimensional manifold (or manifold-with-boundary) in $\mathbf{R}^3$ and let n be the unit outward normal. Then*

$$(1) \qquad dA = n^1\, dy \wedge dz + n^2\, dz \wedge dx + n^3\, dx \wedge dy.$$

Moreover, on M we have

$$(2) \qquad n^1\, dA = dy \wedge dz.$$
$$(3) \qquad n^2\, dA = dz \wedge dx.$$
$$(4) \qquad n^3\, dA = dx \wedge dy.$$

Proof.

Equation (1) is equivalent to the equation

$$dA(v,w) = \det\begin{pmatrix} v \\ w \\ n(x) \end{pmatrix}.$$

This is seen by expanding the determinant by minors along the bottom row. To prove the other equations, let $z \in \mathbf{R}^3{}_x$. Since $v \times w = \alpha n(x)$ for some $\alpha \in \mathbf{R}$, we have

$$\langle z,n(x)\rangle \cdot \langle v \times w,\, n(x)\rangle = \langle z,n(x)\rangle\alpha = \langle z,\alpha n(x)\rangle = \langle z,v \times w\rangle.$$

Choosing $z = e_1$, e_2, and e_3 we obtain (2), (3), and (4). ∎

A word of caution: if $\omega \in \Lambda^2(\mathbf{R}^3{}_a)$ is defined by

$$\begin{aligned}\omega = n^1(a) \cdot dy(a) \wedge dz(a) \\ + n^2(a) \cdot dz(a) \wedge dx(a) \\ + n^3(a) \cdot dx(a) \wedge dy(a),\end{aligned}$$

it is *not* true, for example, that

$$n^1(a) \cdot \omega = dy(a) \wedge dz(a).$$

The two sides give the same result only when applied to $v,w \in M_a$.

A few remarks should be made to justify the definition of length and surface area we have given. If $c\colon [0,1] \to \mathbf{R}^n$ is differentiable and $c([0,1])$ is a one-dimensional manifold-with-boundary, it can be shown, but the proof is messy, that the length of $c([0,1])$ is indeed the least upper bound of the lengths

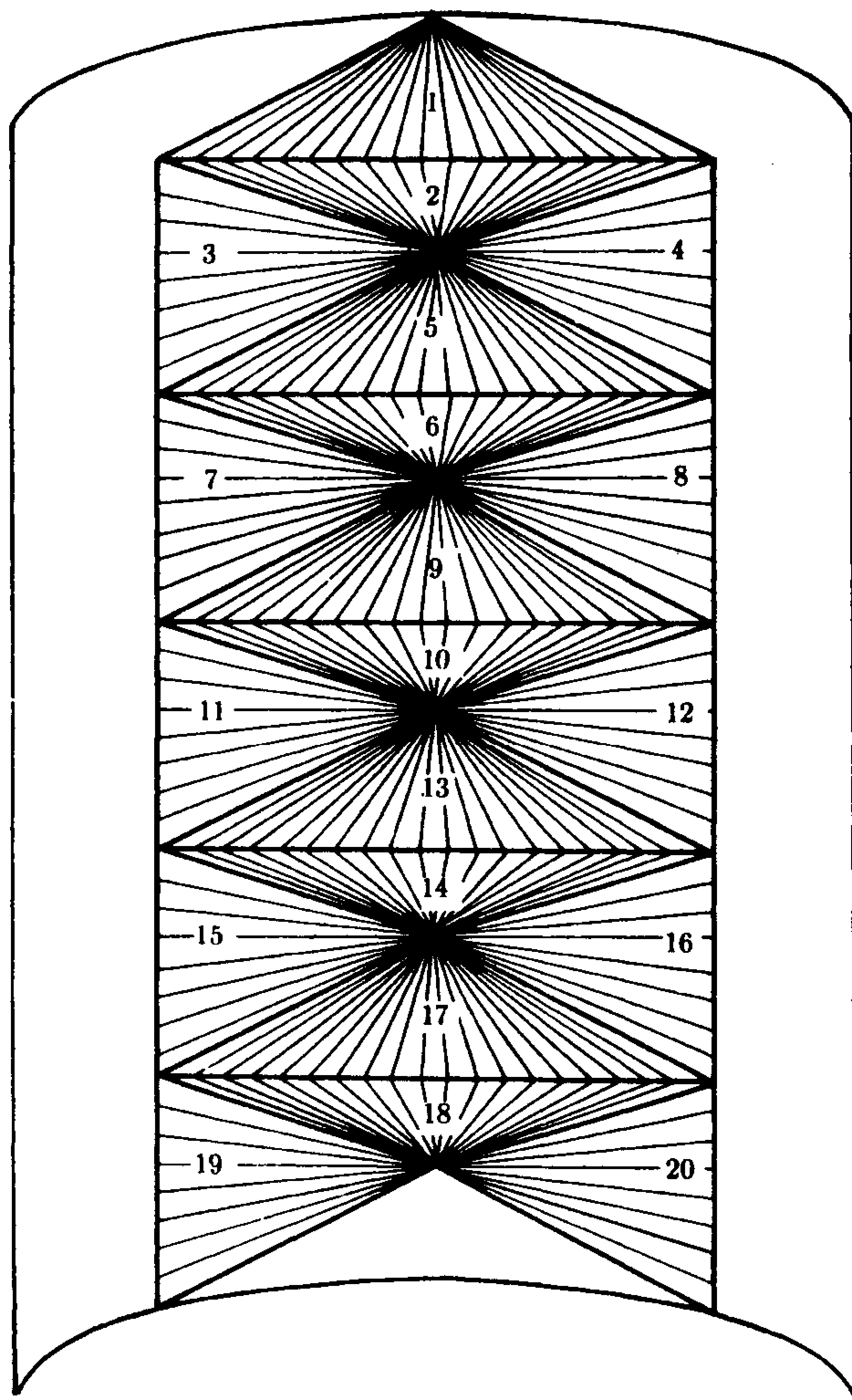


FIGURE 5-9. *A surface containing* 20 *triangles inscribed in a portion of a cylinder. If the number of triangles is increased sufficiently, by making the bases of triangles* 3, 4, 7, 8, *etc., sufficiently small, the total area of the inscribed surface can be made as large as desired.*

of inscribed broken lines. If $c\colon [0,1]^2 \to \mathbf{R}^n$, one naturally hopes that the area of $c([0,1]^2)$ will be the least upper bound of the areas of surfaces made up of triangles whose vertices lie in $c([0,1]^2)$. Amazingly enough, such a least upper bound is usually nonexistent—one can find inscribed polygonal surfaces arbitrarily close to $c([0,1]^2)$ with arbitrarily large area! This is indicated for a cylinder in Figure 5-9. Many definitions of surface area have been proposed, disagreeing with each other, but all agreeing with our definition for differentiable surfaces. For a discussion of these difficult questions the reader is referred to References [3] or [15].

Problems. **5-23.** If M is an oriented one-dimensional manifold in $\mathbf{R}^n$ and $c\colon [0,1] \to M$ is orientation-preserving, show that

$$\int_{[0,1]} c^*(ds) = \int_{[0,1]} \sqrt{[(c^1)']^2 + \cdots + [(c^n)']^2}.$$

5-24. If M is an n-dimensional manifold in $\mathbf{R}^n$, with the usual orientation, show that $dV = dx^1 \wedge \cdots \wedge dx^n$, so that the volume of M, as defined in this section, is the volume as defined in Chapter 3. (Note that this depends on the numerical factor in the definition of $\omega \wedge \eta$.)

5-25. Generalize Theorem 5-6 to the case of an oriented $(n-1)$-dimensional manifold in $\mathbf{R}^n$.

5-26. (a) If $f\colon [a,b] \to \mathbf{R}$ is non-negative and the graph of f in the xy-plane is revolved around the x-axis in $\mathbf{R}^3$ to yield a surface M, show that the area of M is

$$\int_a^b 2\pi f \sqrt{1 + (f')^2}.$$

(b) Compute the area of S^2.

5-27. If $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is a norm preserving linear transformation and M is a k-dimensional manifold in $\mathbf{R}^n$, show that M has the same volume as $T(M)$.

5-28. (a) If M is a k-dimensional manifold, show that an absolute k-tensor $|dV|$ can be defined, even if M is not orientable, so that the volume of M can be defined as $\int_M |dV|$.

(b) If $c\colon [0,2\pi] \times (-1,1) \to \mathbf{R}^3$ is defined by $c(u,v) =$

$$(2\cos u + v\sin(u/2)\cos u,\ 2\sin u + v\sin(u/2)\sin u,\ v\cos u/2),$$

show that $c([0,2\pi] \times (-1,1))$ is a Möbius strip and find its area.

5-29. If there is a nowhere-zero k-form on a k-dimensional manifold M, show that M is orientable.

5-30. (a) If $f\colon [0,1] \to \mathbf{R}$ is differentiable and $c\colon [0,1] \to \mathbf{R}^2$ is defined by $c(x) = (x,f(x))$, show that $c([0,1])$ has length $\int_0^1 \sqrt{1 + (f')^2}$.

(b) Show that this length is the least upper bound of lengths of inscribed broken lines. *Hint:* If $0 = t_0 \le t_1 \le \cdots \le t_n = 1$, then

$$\begin{aligned}|c(t_i) - c(t_{i-1})| &= \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}\\ &= \sqrt{(t_i - t_{i-1})^2 + f'(s_i)^2(t_i - t_{i-1})^2}\end{aligned}$$

for some $s_i \in [t_{i-1},t_i]$.

5-31. Consider the 2-form ω defined on $\mathbf{R}^3 - 0$ by

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

(a) Show that ω is closed.

(b) Show that

$$\omega(p)(v_p,w_p) = \frac{\langle v \times w, p\rangle}{|p|^3}.$$

For $r > 0$ let $S^2(r) = \{x \in \mathbf{R}^3\colon |x| = r\}$. Show that ω restricted to the tangent space of $S^2(r)$ is $1/r^2$ times the volume element, and that $\int_{S^2(r)} \omega = 4\pi$. Conclude that ω is not exact. Nevertheless we denote ω by $d\Theta$ since, as we shall see, $d\Theta$ is the analogue of the 1-form $d\theta$ on $\mathbf{R}^2 - 0$.

(c) If v_p is a tangent vector such that $v = \lambda p$ for some $\lambda \in \mathbf{R}$ show that $d\Theta(p)(v_p,w_p) = 0$ for all w_p. If a two-dimensional manifold M in $\mathbf{R}^3$ is part of a generalized cone, that is, M is the union of segments of rays through the origin, show that $\int_M d\Theta = 0$.

(d) Let $M \subset \mathbf{R}^3 - 0$ be a compact two-dimensional manifold-with-boundary such that every ray through 0 intersects M at most once (Figure 5-10). The union of those rays through 0 which intersect M, is a solid cone $C(M)$. The **solid angle** subtended by M is defined as the area of $C(M) \cap S^2$, or equivalently as $1/r^2$ times the area of $C(M) \cap S^2(r)$ for $r > 0$. Prove that the solid angle subtended by M is $\left|\int_M d\Theta\right|$. *Hint:* Choose r small enough so that there is a three-dimensional manifold-with-boundary N (as in Figure 5-10) such that ∂N is the union of M and $C(M) \cap S^2(r)$, and a part of a generalized cone. (Actually, N will be a manifold-with-corners; see the remarks at the end of the next section.)

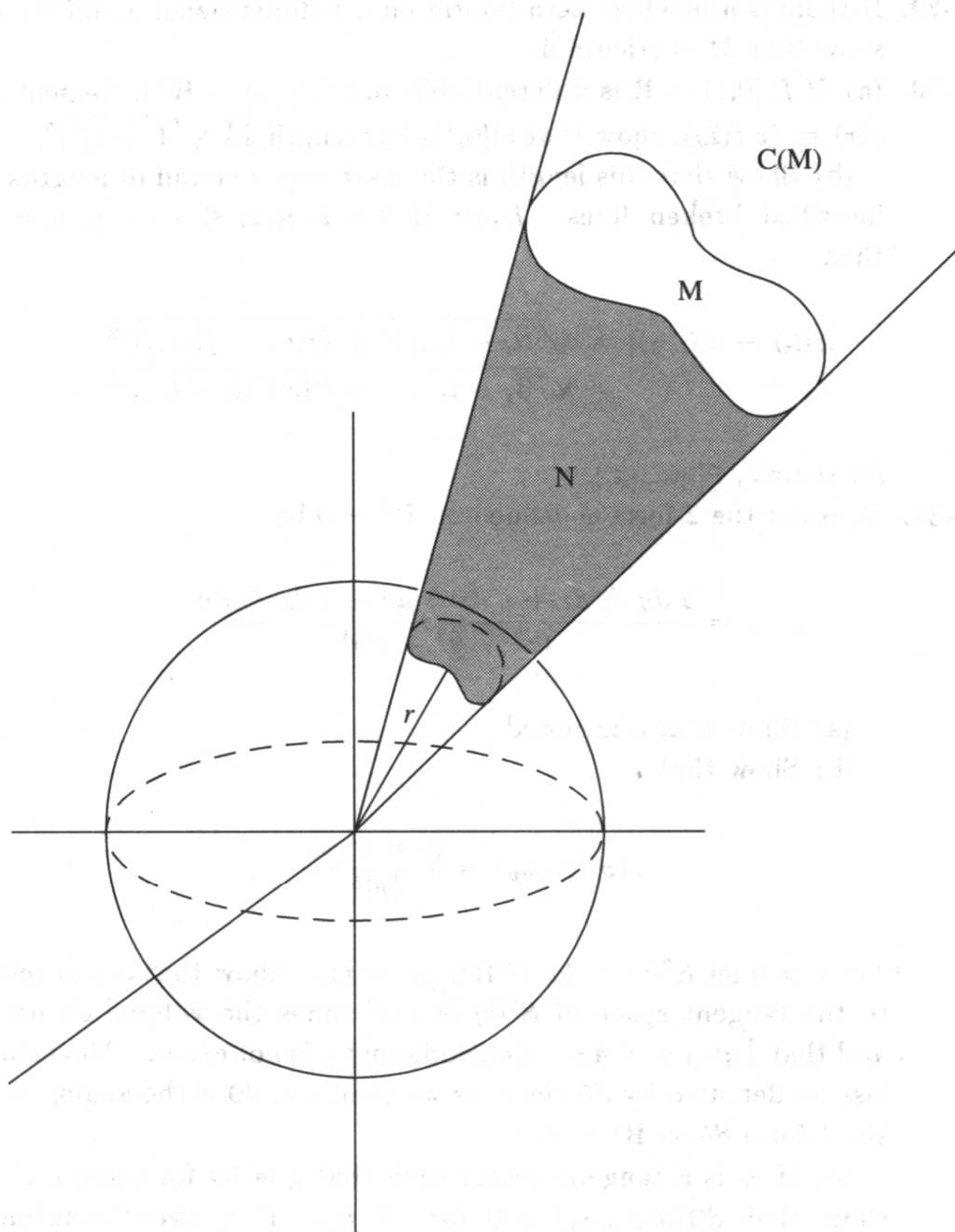


FIGURE 5-10

5-32. Let f, g: $[0,1] \to \mathbf{R}^3$ be nonintersecting closed curves. Define the *linking number* $l(f,g)$ of f and g by (*cf.* Problem 4-34)

$$l(f,g) = \frac{-1}{4\pi} \int_{c_{f,g}} d\Theta.$$

(a) Show that if (F,G) is a homotopy of nonintersecting closed curves, then $l(F_0,G_0) = l(F_1,G_1)$.

(b) If $r(u,v) = |f(u) - g(v)|$ show that

$$l(f,g) = \frac{-1}{4\pi}\int_0^1\int_0^1 \frac{1}{[r(u,v)]^3}\cdot A(u,v)\,du\,dv$$

where

$$A(u,v) = \det\begin{pmatrix} (f^1)'(u) & (f^2)'(u) & (f^3)'(u) \\ (g^1)'(v) & (g^2)'(v) & (g^3)'(v) \\ f^1(u) - g^1(v) & f^2(u) - g^2(v) & f^3(u) - g^3(v) \end{pmatrix}.$$

(c) Show that $l(f,g) = 0$ if f and g both lie in the xy-plane. The curves of Figure 4-5 (b) are given by $f(u) = (\cos u, \sin u, 0)$ and $g(v) = (1 + \cos v,\ 0,\ \sin v)$. You may easily convince yourself that calculating $l(f,g)$ by the above integral is hopeless in this case. The following problem shows how to find $l(f,g)$ without explicit calculations.

5-33. (a) If $(a,b,c) \in \mathbf{R}^3$ define

$$d\Theta_{(a,b,c)} = \frac{(x-a)dy \wedge dz + (y-b)dz \wedge dx + (z-c)dx \wedge dy}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{\frac{3}{2}}}.$$

If M is a compact two-dimensional manifold-with-boundary in $\mathbf{R}^3$ and $(a,b,c) \notin M$ define

$$\Omega(a,b,c) = \int_M d\Theta_{(a,b,c)}.$$

Let (a,b,c) be a point on the same side of M as the outward normal and (a',b',c') a point on the opposite side. Show that by choosing (a,b,c) sufficiently close to (a',b',c') we can make $\Omega(a,b,c) - \Omega(a',b',c')$ as close to -4π as desired. *Hint:* First show that if $M = \partial N$ then $\Omega(a,b,c) = -4\pi$ for $(a,b,c) \in N - M$ and $\Omega(a,b,c) = 0$ for $(a,b,c) \notin N$.

(b) Suppose $f([0,1]) = \partial M$ for some compact oriented two-dimensional manifold-with-boundary M. (If f does not intersect itself such an M always exists, even if f is knotted, see [6], page 138.) Suppose that whenever g intersects M at x the tangent vector v of g is not in M_x. Let n^+ be the number of intersections where v points in the same direction as the outward normal and n^- the number of other intersections. If $n = n^+ - n^-$ show that

$$n = \frac{-1}{4\pi}\int_g d\Omega.$$

(c) Prove that

$$D_1\Omega(a,b,c) = \int_f \frac{(y-b)dz - (z-c)dy}{r^3}$$

$$D_2\Omega(a,b,c) = \int_f \frac{(z-c)dx - (x-a)dz}{r^3}$$

$$D_3\Omega(a,b,c) = \int_f \frac{(x-a)dy - (y-b)dx}{r^3},$$

where $r(x,y,z) = |(x,y,z)|$.

(d) Show that the integer n of (b) equals the integral of Problem 5-32(b), and use this result to show that $l(f,g) = 1$ if f and g are the curves of Figure 4-6 (b), while $l(f,g) = 0$ if f and g are the curves of Figure 4-6 (c). (These results were known to Gauss [7]. The proofs outlined here are from [4] pp. 409–411; see also [13], Volume 2, pp. 41–43.)

THE CLASSICAL THEOREMS

We have now prepared all the machinery necessary to state and prove the classical "Stokes' type" of theorems. We will indulge in a little bit of self-explanatory classical notation.

5-7 Theorem (Green's Theorem). *Let $M \subset \mathbf{R}^2$ be a compact two-dimensional manifold-with-boundary. Suppose that $\alpha,\beta\colon M \to \mathbf{R}$ are differentiable. Then*

$$\int_{\partial M} \alpha\, dx + \beta\, dy = \int_M (D_1\beta - D_2\alpha)dx \wedge dy$$
$$= \iint_M \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx\, dy.$$

(Here M is given the usual orientation, and ∂M the induced orientation, also known as the counterclockwise orientation.)

Proof. This is a very special case of Theorem 5-5, since $d(\alpha\, dx + \beta\, dy) = (D_1\beta - D_2\alpha)dx \wedge dy$. ▌

5-8 Theorem (Divergence Theorem). *Let $M \subset \mathbf{R}^3$ be a compact three-dimensional manifold-with-boundary and n the unit outward normal on ∂M. Let F be a differentiable vector field on M. Then*

$$\int_M \operatorname{div} F\, dV = \int_{\partial M} \langle F,n\rangle\, dA.$$

This equation is also written in terms of three differentiable functions $\alpha,\beta,\gamma\colon M \to \mathbf{R}$:

$$\iiint_M \left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z}\right) dV = \iint_{\partial M} (n^1\alpha + n^2\beta + n^3\gamma)\, dS.$$

Proof. Define ω on M by $\omega = F^1\, dy \wedge dz + F^2\, dz \wedge dx + F^3\, dx \wedge dy$. Then $d\omega = \operatorname{div} F\, dV$. According to Theorem 5-6, on ∂M we have

$$\begin{aligned} n^1\, dA &= dy \wedge dz,\\ n^2\, dA &= dz \wedge dx,\\ n^3\, dA &= dx \wedge dy. \end{aligned}$$

Therefore on ∂M we have

$$\begin{aligned} \langle F,n\rangle\, dA &= F^1 n^1\, dA + F^2 n^2\, dA + F^3 n^3\, dA\\ &= F^1\, dy \wedge dz + F^2\, dz \wedge dx + F^3\, dx \wedge dy\\ &= \omega. \end{aligned}$$

Thus, by Theorem 5-5 we have

$$\int_M \operatorname{div} F\, dV = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} \langle F, n\rangle\, dA. \quad \blacksquare$$

5-9 Theorem (Stokes' Theorem). *Let $M \subset \mathbf{R}^3$ be a compact oriented two-dimensional manifold-with-boundary and n the unit outward normal on M determined by the orientation of M. Let ∂M have the induced orientation. Let T be the vector field on ∂M with $ds(T) = 1$ and let F be a differentiable vector field in an open set containing M. Then*

$$\int_M \langle(\nabla \times F), n\rangle\, dA = \int_{\partial M} \langle F,T\rangle\, ds.$$

This equation is sometimes written

$$\int_{\partial M} \alpha\, dx + \beta\, dy + \gamma\, dz =$$

$$\iint_M \left[n^1 \left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) + n^2 \left(\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right) + n^3 \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \right] dS.$$

Proof. Define ω on M by $\omega = F^1\, dx + F^2\, dy + F^3\, dz$. Since $\nabla \times F$ has components $D_2F^3 - D_3F^2$, $D_3F^1 - D_1F^3$, $D_1F^2 - D_2F^1$, it follows, as in the proof of Theorem 5-8, that on M we have

$$\begin{aligned} \langle(\nabla \times F), n\rangle\, dA &= (D_2F^3 - D_3F^2) dy \wedge dz \\ &\qquad + (D_3F^1 - D_1F^3) dz \wedge dx \\ &\qquad\qquad + (D_1F^2 - D_2F^1) dx \wedge dy \\ &= d\omega. \end{aligned}$$

On the other hand, since $ds(T) = 1$, on ∂M we have

$$\begin{aligned} T^1\, ds &= dx, \\ T^2\, ds &= dy, \\ T^3\, ds &= dz. \end{aligned}$$

(These equations may be checked by applying both sides to T_x, for $x \in \partial M$, since T_x is a basis for $(\partial M)_x$.)

Therefore on ∂M we have

$$\begin{aligned} \langle F,T\rangle\, ds &= F^1T^1\, ds + F^2T^2\, ds + F^3T^3\, ds \\ &= F^1\, dx + F^2\, dy + F^3\, dz \\ &= \omega. \end{aligned}$$

Thus, by Theorem 5-5, we have

$$\int_M \langle(\nabla \times F),n\rangle\, dA = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} \langle F,T\rangle\, ds. \quad \blacksquare$$

Theorems 5-8 and 5-9 are the basis for the names div F and curl F. If $F(x)$ is the velocity vector of a fluid at x (at some time) then $\int_{\partial M} \langle F,n\rangle\, dA$ is the amount of fluid "diverging" from M. Consequently the condition div $F = 0$ expresses

the fact that the fluid is incompressible. If M is a disc, then $\int_{\partial M} \langle F,T \rangle \, ds$ measures the amount that the fluid curls around the center of the disc. If this is zero for all discs, then $\nabla \times F = 0$, and the fluid is called *irrotational.*

These interpretations of div F and curl F are due to Maxwell [13]. Maxwell actually worked with the negative of div F, which he accordingly called the *convergence.* For $\nabla \times F$ Maxwell proposed "with great diffidence" the terminology *rotation of F;* this unfortunate term suggested the abbreviation rot F which one occasionally still sees.

The classical theorems of this section are usually stated in somewhat greater generality than they are here. For example, Green's Theorem is true for a square, and the Divergence Theorem is true for a cube. These two particular facts can be proved by approximating the square or cube by manifolds-with-boundary. A thorough generalization of the theorems of this section requires the concept of manifolds-with-corners; these are subsets of $\mathbf{R}^n$ which are, up to diffeomorphism, locally a portion of $\mathbf{R}^k$ which is bounded by pieces of $(k-1)$-planes. The ambitious reader will find it a challenging exercise to define manifolds-with-corners rigorously and to investigate how the results of this entire chapter may be generalized.

Problems. **5-34.** Generalize the divergence theorem to the case of an n-manifold with boundary in $\mathbf{R}^n$.

5-35. Applying the generalized divergence theorem to the set $M = \{x \in \mathbf{R}^n\colon |x| \le a\}$ and $F(x) = x_x$, find the volume of $S^{n-1} = \{x \in \mathbf{R}^n\colon |x| = 1\}$ in terms of the n-dimensional volume of $B_n = \{x \in \mathbf{R}^n\colon |x| \le 1\}$. (This volume is $\pi^{n/2}/(n/2)!$ if n is even and $2^{(n+1)/2}\pi^{(n-1)/2}/1 \cdot 3 \cdot 5 \cdot \ldots \cdot n$ if n is odd.)

5-36. Define F on $\mathbf{R}^3$ by $F(x) = (0,0,cx^3)_x$ and let M be a compact three-dimensional manifold-with-boundary with $M \subset \{x\colon x^3 \le 0\}$. The vector field F may be thought of as the downward pressure of a fluid of density c in $\{x\colon x^3 \le 0\}$. Since a fluid exerts equal pressures in all directions, we define the *buoyant force* on M, due to the fluid, as $-\int_{\partial M} \langle F,n \rangle \, dA$. Prove the following theorem.

Theorem (*Archimedes*). The buoyant force on M is equal to the weight of the fluid displaced by M.

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Index

Addenda

1. It should be remarked after Theorem 2-11 (the Inverse Function Theorem) that the formula for f^{-1} allows us to conclude that f^{-1} is actually continuously differentiable (and that it is C^∞ if f is). Indeed, it suffices to note that the entries of the inverse of a matrix A are C^∞ functions of the entries of A. This follows from "Cramer's Rule": $(A^{-1})_{ji} = (\det A^{ij})/(\det A)$, where A^{ij} is the matrix obtained from A by deleting row i and column j.

2. The proof of the first part of Theorem 3-8 can be simplified considerably, rendering Lemma 3-7 unnecessary. It suffices to cover B by the interiors of closed rectangles U_i with $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$, and to choose for each $x \in A - B$ a closed rectangle V_x, containing x in its interior, with $M_{V_x}(f) - m_{V_x}(f) < \varepsilon$. If every subrectangle of a partition P is contained in one of some finite collection of U_i's and V_x's which cover A, and $|f(x)| \leq M$ for all x in A, then $U(f, P) - L(f, P) < \varepsilon v(A) + 2M\varepsilon$.

The proof of the converse part contains an error, since $M_S(f) - m_S(f) \geq 1/n$ is guaranteed only if the interior of S intersects $B_{1/n}$. To compensate for this it suffices to cover the boundaries of all subrectangles of P with a finite collection of rectangles with total volume $< \varepsilon$. These, together with $\mathcal{S}$, cover $B_{1/n}$, and have total volume $< 2\varepsilon$.

3. The argument in the first part of Theorem 3-14 (Sard's Theorem) requires a little amplification. If $U \subset A$ is a closed rectangle with sides of length l, then, because U is compact, there is an integer N with the following property: if U is divided into N^n rectangles, with sides of length l/N, then $|D_j g^i(w) - D_j g^i(z)| < \varepsilon/n^2$ whenever w and z are both in one such rectangle S. Given $x \in S$, let $f(z) = Dg(x)(z) - g(z)$. Then, if $z \in S$,

$$|D_j f^i(z)| = |D_j g^i(x) - D_j g^i(z)| < \varepsilon/n^2.$$

So by Lemma 2-10, if $x,y \in S$, then

$$\begin{aligned}|Dg(x)(y - x) - g(y) + g(x)| = |f(y) - f(x)| &< \varepsilon|x - y| \\ &\leq \varepsilon\sqrt{n}\,(l/N).\end{aligned}$$

4. Finally, the notation $\Lambda^k(V)$ appearing in this book is incorrect, since it conflicts with the standard definition of $\Lambda^k(V)$ (as a certain quotient of the tensor algebra of V). For the vector space in question (which is naturally isomorphic to $\Lambda^k(V^*)$ for finite dimensional vector spaces V) the notation $\Omega^k(V)$ is probably on the way to becoming standard. This substitution should be made on pages 78–85, 88–89, 116, and 126–128.

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